Extended supersymmetric sigma models in AdS$_4$ from projective superspace

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Abstract

There exist two superspace approaches to describe $\mathcal{N} = 2$ supersymmetric non-linear σ-models in four-dimensional anti-de Sitter (AdS$_4$) space: (i) in terms of $\mathcal{N} = 1$ AdS chiral superfields, as developed in arXiv:1105.3111 and arXiv:1108.5290; and (ii) in terms of $\mathcal{N} = 2$ polar supermultiplets using the AdS projective-superspace techniques developed in arXiv:0807.3368. The virtue of the approach (i) is that it makes manifest the geometric properties of the $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$. The target space must be a non-compact hyperkähler manifold endowed with a Killing vector field which generates an SO(2) group of rotations on the two-sphere of complex structures. The power of the approach (ii) is that it allows us, in principle, to generate hyperkähler metrics as well as to address the problem of deformations of such metrics.

Here we show how to relate the formulation (ii) to (i) by integrating out an infinite number of $\mathcal{N} = 1$ AdS auxiliary superfields and performing a superfield duality transformation. We also develop a novel description of the most general $\mathcal{N} = 2$ supersymmetric nonlinear σ-model in AdS$_4$ in terms of chiral superfields on three-dimensional $\mathcal{N} = 2$ flat superspace without central charge. This superspace naturally originates from a conformally flat realization for the four-dimensional $\mathcal{N} = 2$ AdS superspace that makes use of Poincaré coordinates for AdS$_4$. This novel formulation allows us to uncover several interesting geometric results.

Contents

1 Introduction

Recently, two of us (DB and SMK) have constructed the most general $\mathcal{N} = 2$ super-symmetric σ-model in four-dimensional anti-de Sitter space (AdS$_4$) using a formulation in terms of $\mathcal{N} = 1$ covariantly chiral superfields [1, 2]. The target space of such a σ-model

proves to be a non-compact hyperkähler manifold restricted to possess a special Killing vector field which generates an SO(2) group of rotations on the two-sphere of complex structures and necessarily leaves one of them, $\mathbb{J}$, invariant; each of the complex structures that are orthogonal to $\mathbb{J}$ is characterized by an exact Kähler two-form.[1] The existence of such hyperkähler spaces was pointed out twenty five years ago in [4]. One of the main virtues of the $\mathcal{N} = 1$ formulation [1, 2] is its geometric character. The superfield Lagrangian proves to be a globally defined function on the target space, which is simultaneously the Kähler potential (with respect to each complex structure orthogonal to $\mathbb{J}$) and the Killing potential of the SO(2) isometry group (with respect to $\mathbb{J}$). Another remarkable property of the $\mathcal{N} = 2$ supersymmetric σ-model constructed in [1, 2] is that the algebra of OSp(2|4) transformations closes off the mass shell. The only disadvantage of the $\mathcal{N} = 1$ formulation [1, 2] is that it cannot be used to generate hyperkähler metrics (a hyperkähler space has to be given in order to define the σ-model action). The latter goal can be achieved by resorting to powerful $\mathcal{N} = 2$ superspace techniques such as harmonic superspace [5, 6] and projective superspace [7, 8, 9].

A few years ago, general off-shell $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ were constructed in $\mathcal{N} = 2$ AdS superspace [10] building on the projective-superspace formulations for $\mathcal{N} = 2$ matter-coupled supergravity in four dimensions [11, 12] and $\mathcal{N} = 1$ matter-coupled supergravity in five dimensions [13, 14]. The work of [10] is a natural extension of the earlier 5D AdS approach developed in [15, 16]. The powerful property of the construction given in [10] is that $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ can be generated from a Lagrangian that is an arbitrary real analytic function of $2n$ real variables, where $4n$ is the dimension of the target space. Its technical disadvantage is that the hyperkähler geometry of the target space is hidden, unlike in the $\mathcal{N} = 1$ formulation [1, 2]. To uncover the explicit structure of the target space, the $\mathcal{N} = 2$ formulation of [10] has to be related to that given [1, 2]. For the series of $\mathcal{N} = 2$ supersymmetric σ-models presented in [10], one can in principle derive their reformulation in terms of $\mathcal{N} = 1$ chiral superfields by:

(i) eliminating the (infinitely many) $\mathcal{N} = 1$ auxiliary superfields; and

(ii) performing appropriate $\mathcal{N} = 1$ superfield duality transformations.

These are nontrivial technical problems which are more difficult to address than in Minkowski space, due to non-zero curvature of the AdS space-time. These problems will be dealt with in the present paper. Before turning to the description of our novel approach, it is appropriate to recall the salient points of [1, 2] and [10].

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[1]In the case of 4D $\mathcal{N} = 2$ Poincaré supersymmetry, the target-space geometry of general σ-models is only required to be hyperkähler [3].

1.1 $\mathcal{N} = 2$ anti-de Sitter superspace

In order to make use of the power of projective superspace techniques [7, 8, 9], one has to pick an $\mathcal{N} = 1$ subspace of a given $\mathcal{N} = 2$ superspace, which in our case is $\mathcal{N} = 2$ AdS superspace. It is pertinent here to recall its definition. The four-dimensional $\mathcal{N} = 2$ AdS superspace

$$\text{AdS}^{4|8} := \frac{\text{OSp}(2|4)}{\text{SO}(3, 1) \times \text{SO}(2)}$$

is a maximally symmetric geometry that originates within the superspace formulation of $\mathcal{N} = 2$ conformal supergravity developed in [11]. The corresponding covariant derivatives[2]

$$\mathcal{D}_{\mathcal{A}} = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_i^{\dot{\alpha}}) = E_{\mathcal{A}}{}^{\mathcal{M}} \partial_{\mathcal{M}} + \frac{1}{2} \Omega_{\mathcal{A}}{}^{bc} M_{bc} + \Phi_{\mathcal{A}}{}^{ij} J_{ij} , \qquad i, j = \underline{1}, \underline{2} \tag{1.1}$$

obey the algebra [10, 11]

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = 4\mathcal{S}^{ij} M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\mathcal{S}^{kl} J_{kl} , \qquad \{\mathcal{D}_\alpha^i, \bar{\mathcal{D}}_j^{\dot{\beta}}\} = -2\mathrm{i}\delta_j^i (\sigma^c)_\alpha{}^{\dot{\beta}} \mathcal{D}_c , \tag{1.2a}$$

$$[\mathcal{D}_a, \mathcal{D}_\beta^j] = \tfrac{\mathrm{i}}{2} (\sigma_a)_{\beta\dot{\gamma}} \mathcal{S}^{jk} \bar{\mathcal{D}}_k^{\dot{\gamma}} , \qquad [\mathcal{D}_a, \bar{\mathcal{D}}_j^{\dot{\beta}}] = \tfrac{\mathrm{i}}{2} (\tilde{\sigma}_a)^{\dot{\beta}\gamma} \mathcal{S}_{jk} \mathcal{D}_\gamma^k , \tag{1.2b}$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -\mathcal{S}^2 M_{ab} , \tag{1.2c}$$

where $\mathcal{S}^{ij}$ is a *covariantly constant* real isotriplet, $\mathcal{D}_{\mathcal{A}} \mathcal{S}^{ij} = 0$, with the algebraic properties $\mathcal{S}^{ji} = \mathcal{S}^{ij}$, $\overline{\mathcal{S}^{ij}} = \mathcal{S}_{ij} = \varepsilon_{ik}\varepsilon_{jl}\mathcal{S}^{kl}$, and $\mathcal{S}^2 := \frac{1}{2}\mathcal{S}^{ij}\mathcal{S}_{ij} = \text{const}$. The constant $\mathcal{S}^2$ is positive and so (1.2c) gives the algebra of covariant derivatives in AdS. This superspace is conformally flat [10] and proves to be a solution to the equations of motion for $\mathcal{N} = 2$ supergravity with a cosmological term [17].

Due to (1.2), the SU(2) gauge freedom can be used to choose the SU(2) connection $\Phi_{\mathcal{A}}{}^{ij}$ in (1.1) to look like $\Phi_{\mathcal{A}}{}^{ij} = \Phi_{\mathcal{A}} \mathcal{S}^{ij}$, for some one-form $\Phi_{\mathcal{A}}$ describing the residual U(1) connection associated with the generator $\mathcal{S}^{ij} J_{ij}$. Then $\mathcal{S}^{ij}$ becomes a constant isotriplet, $\mathcal{S}^{ij} = s^{ij} = \text{const}$. The remaining global SU(2) rotations can take s^{ij} to any position on the two-sphere of radius $s = \sqrt{\frac{1}{2} s^{ij} s_{ij}} \equiv \mathcal{S}$.[3] There are two natural options for how to choose $\mathcal{S}^{ij}$:

$$s^{\underline{12}} = 0 ; \tag{1.3a}$$

$$s^{\underline{11}} = s^{\underline{22}} = 0 . \tag{1.3b}$$

Of course, these options are physically equivalent. However, choosing one or the other may be more preferable to achieve certain technical simplifications. It turns out that the

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[2]The SU(2) generators, J_{kl}, act on the spinor derivatives by the rule: $[J_{kl}, \mathcal{D}_\alpha^i] = -\frac{1}{2}(\delta_k^i \mathcal{D}_{\alpha l} + \delta_l^i \mathcal{D}_{\alpha k})$.
[3]In what follows, we do not distinguish between s and $\mathcal{S}$.

choice (1.3a) must be used in order to embed an $\mathcal{N} = 1$ AdS superspace, AdS$^{4|4}$, into the full $\mathcal{N} = 2$ AdS superspace [10]. As to the second choice, eq. (1.3b), it will be shown in this paper that it corresponds to choosing Poincaré coordinates[4] for AdS$_4$ in which the space-time metric takes the form

$$\mathrm{d}s^2 = \left(\frac{1}{sz}\right)^2 \left(\eta_{mn}\mathrm{d}x^m\mathrm{d}x^n + \mathrm{d}z^2\right) . \tag{1.4}$$

The slices $z = \mathrm{const}$ foliate AdS$_4$ into a family of three-dimensional Minkowski spaces. We will show, closely following the 5D AdS construction of [16], that the choice (1.3b) allows us[5] to choose a different $\mathcal{N} = 1$ subspace of AdS$^{4|8}$, specifically $\mathbb{R}^{3|4} \times \mathbb{R}_+$, where $\mathbb{R}^{3|4}$ denotes three-dimensional $\mathcal{N} = 2$ Minkowski superspace *without* central charge, and $\mathbb{R}_+ := \{z \in \mathbb{R}, \ z > 0\}$. Using such a setting, the problem of reformulating the off-shell $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ [10] in terms of $\mathcal{N} = 1$ chiral superfields proves to become almost identical to that appearing in the case of off-shell $\mathcal{N} = 2$ supersymmetric σ-models in 4D $\mathcal{N} = 2$ Minkowski space. The latter problem has been addressed in a number of publications [18, 19, 20, 21, 22, 23] (see [24] for a review), and here we can make use of the results obtained in these papers.

1.2   Formulation of $\mathcal{N} = 2$ supersymmetric σ-models in AdS$^{4|4}$

As already mentioned, the choice (1.3a) is required for embedding AdS$^{4|4}$ into AdS$^{4|8}$. We assume that AdS$^{4|8}$ is parametrized by local bosonic (x) and fermionic $(\theta, \bar{\theta})$ coordinates $\boldsymbol{z}^{\mathcal{M}} = (x^m, \theta^\mu_\imath, \bar{\theta}^{\dot{\imath}}_{\dot{\mu}})$ (where $m = 0, 1, \cdots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $\imath = \underline{1}, \underline{2}$). By applying certain general coordinate and local U(1) transformations in AdS$^{4|4}$, it is possible to identify AdS$^{4|4}$ with the surface $\theta^\mu_{\underline{2}} = 0$ and $\bar{\theta}^{\underline{2}}_{\dot{\mu}} = 0$. The covariant derivatives for AdS$^{4|4}$,

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A{}^M \partial_M + \frac{1}{2}\Omega_A{}^{bc}M_{bc} , \tag{1.5}$$

are related to (1.1) as follows

$$\mathcal{D}_\alpha := \mathcal{D}^{\underline{1}}_\alpha\big| , \qquad \bar{\mathcal{D}}^{\dot{\alpha}} := \bar{\mathcal{D}}^{\dot{\alpha}}_{\underline{1}}\big| , \tag{1.6}$$

and similarly for the vector covariant derivative. Here the bar-projection is defined by

$$U\big| := U(x, \theta_\imath, \bar{\theta}^{\dot{\imath}})\big|_{\theta_{\underline{2}} = \bar{\theta}^{\underline{2}} = 0} , \tag{1.7}$$

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[4]The Poincaré patch covers half of AdS$_4$. It is sufficient to restrict our analysis to this coordinate patch when considering infinitesimal isometry transformations.

[5]More precisely, we will use a conformally flat representation for the covariant derivatives (1.1) such that $\mathcal{S}^{ij}$ is not constant but instead $\mathcal{S}^{ij} = s^{ij} + O(\theta)$. Then, the choice (1.3b) leads to the required 3D foliation.

for any $\mathcal{N} = 2$ tensor superfield $U(x, \theta_i, \bar{\theta}^i)$. It follows from (1.2) that the $\mathcal{N} = 1$ covariant derivatives obey the algebra

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{\mu} M_{\alpha\beta} \ , \quad \{\bar{\mathcal{D}}_{\dot\alpha}, \bar{\mathcal{D}}_{\dot\beta}\} = 4\mu \bar{M}_{\dot\alpha\dot\beta} \ , \quad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} = -2\mathrm{i}\mathcal{D}_{\alpha\dot\beta} \ , \tag{1.8a}$$

$$[\mathcal{D}_a, \mathcal{D}_\beta] = -\tfrac{\mathrm{i}}{2}\bar{\mu}(\sigma_a)_{\beta\dot\gamma}\bar{\mathcal{D}}^{\dot\gamma} \ , \quad [\mathcal{D}_a, \bar{\mathcal{D}}_{\dot\beta}] = \tfrac{\mathrm{i}}{2}\mu(\sigma_a)_{\gamma\dot\beta}\mathcal{D}^\gamma \ , \tag{1.8b}$$

$$[\mathcal{D}_a, \mathcal{D}_b] = -|\mu|^2 M_{ab} \ , \tag{1.8c}$$

where $\mu = -s_{\underline{11}} = -s^{\underline{22}}$. As a result, each $\mathcal{N} = 2$ supersymmetric field theory in AdS$^{4|8}$ can be reformulated as some theory in AdS$^{4|4}$.

Any $\mathcal{N} = 2$ supersymmetric nonlinear σ-model describes a self-interaction of hypermultiplets. When formulated in $\mathcal{N} = 1$ AdS superspace, a single hypermultiplet can be realized in terms of two covariantly chiral scalar superfields. As shown in [1, 2], the most general $\mathcal{N} = 2$ supersymmetric nonlinear σ-model in AdS can be described by an action in AdS$^{4|4}$ of the form[6]

$$S = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, E\, \mathcal{K}(\varphi^a, \bar{\varphi}^{\bar{b}}) \ , \qquad E^{-1} = \mathrm{Ber}(E_A{}^M) \tag{1.9}$$

where φ^a is a chiral scalar, $\bar{\mathcal{D}}_{\dot\alpha}\varphi^a = 0$. Here $\mathcal{K}(\varphi, \bar{\varphi})$ is a globally defined real function over the target space $\mathcal{M}$ which is a hyperkähler manifold. In terms of $\mathcal{K}(\varphi, \bar{\varphi})$, the target space metric is $g_{a\bar{b}} = \partial_a\partial_{\bar{b}}\mathcal{K}$, and hence the Kähler two-form is exact. This implies that the target space is non-compact. The variables φ^a are local complex coordinates with respect to one of the complex structures on $\mathcal{M}$,

$$J_3 = \begin{pmatrix} \mathrm{i}\,\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix} \ . \tag{1.10}$$

Two other complex structures can be chosen as

$$J_1 = \begin{pmatrix} 0 & \omega^a{}_{\bar{b}} \\ \omega^{\bar{a}}{}_b & 0 \end{pmatrix} \ , \qquad J_2 = \begin{pmatrix} 0 & \mathrm{i}\,\omega^a{}_{\bar{b}} \\ -\mathrm{i}\,\omega^{\bar{a}}{}_b & 0 \end{pmatrix} \ , \tag{1.11}$$

and $\mathcal{M}$ is Kähler with respect to each of them. Here $\omega_{ab} := g_{a\bar{c}}\omega^{\bar{c}}{}_b = -\omega_{ba}$ is a covariantly constant (2,0) form with respect to J_3,

$$\nabla_c\omega_{ab} = \nabla_{\bar{c}}\omega_{ab} = 0 \ , \tag{1.12}$$

and hence it is holomorphic, $\omega_{ab} = \omega_{ab}(\varphi)$. The operators $J_A = (J_1, J_2, J_3)$ obey the quaternionic algebra $J_A J_B = -\delta_{AB}\mathbb{1} + \varepsilon_{ABC}J_C$.

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[6]The target space of the most general $\mathcal{N} = 1$ supersymmetric σ-model in AdS (1.9) is characterized by an exact Kähler two-form [25, 26, 1], and therefore this manifold is non-compact.

As shown in [1, 2], the σ-model (1.9) is $\mathcal{N} = 2$ supersymmetric provided the following vector field

$$V^\nu = (V^a, V^{\bar{a}}) = \left(\frac{\mu}{2|\mu|} \omega^{ab} \mathcal{K}_b \, , \, \frac{\bar{\mu}}{2|\mu|} \omega^{\bar{a}\bar{b}} \mathcal{K}_{\bar{b}} \right) \tag{1.13}$$

obeys the Killing equations[7]

$$\nabla_a V_b + \nabla_b V_a = \nabla_a V_{\bar{b}} + \nabla_{\bar{b}} V_a = 0 \ . \tag{1.14}$$

It can be shown that this Killing vector field *rotates* the complex structures:

$$\mathcal{L}_V J_1 = J_3 \sin\theta \, , \quad \mathcal{L}_V J_2 = -J_3 \cos\theta \, , \quad \mathcal{L}_V J_3 = J_2 \cos\theta - J_1 \sin\theta \, , \tag{1.15}$$

where $\theta := \arg\mu$. There is a preferred complex structure

$$\mathbb{J} := J_1 \cos\theta + J_2 \sin\theta = \frac{1}{|\mu|} \begin{pmatrix} 0 & \mu\,\omega^a{}_{\bar{b}} \\ \bar{\mu}\,\omega^{\bar{a}}{}_b & 0 \end{pmatrix} \tag{1.16}$$

with respect to which V^ν is *holomorphic*,

$$\mathcal{L}_V \mathbb{J} = 0 \ . \tag{1.17}$$

1.3 Formulation of $\mathcal{N} = 2$ supersymmetric σ-models in AdS$^{4|8}$

General supersymmetric field theories in AdS$^{4|8}$ can be formulated in terms of covariant projective supermultiplets [10]. The covariant projective supermultiplets in four-dimensional $\mathcal{N} = 2$ supergravity were introduced in [11]. The definition given in [11] was then specialized to the case of $\mathcal{N} = 2$ AdS supersymmetry in [10]. A projective supermultiplet of weight n, $\mathcal{Q}^{(n)}(v^i)$, is defined to be a scalar superfield that lives on AdS$^{4|8}$, is holomorphic with respect to the isotwistor variables v^i on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(1) it obeys the covariant analyticity constraints

$$\mathcal{D}_\alpha^{(1)} \mathcal{Q}^{(n)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(1)} \mathcal{Q}^{(n)} = 0 \, , \qquad \mathcal{D}_\alpha^{(1)} := v_i \mathcal{D}_\alpha^i \, , \quad \bar{\mathcal{D}}_{\dot{\alpha}}^{(1)} := v_i \bar{\mathcal{D}}_{\dot{\alpha}}^i \ ; \tag{1.18}$$

(2) it is a homogeneous function of v^i of degree n, that is,

$$\mathcal{Q}^{(n)}(c\,v) = c^n \, \mathcal{Q}^{(n)}(v) \, , \qquad c \in \mathbb{C} \setminus \{0\} \ ; \tag{1.19}$$

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[7]The equation $\nabla_a V_b + \nabla_b V_a = 0$ trivially follows from the definition (1.13).

(3) the OSp(2|4) transformation law of $\mathcal{Q}^{(n)}$ is as follows:

$$\delta_\xi \mathcal{Q}^{(n)} = -\left(\xi + 2\varepsilon \mathcal{S}^{ij} J_{ij}\right) \mathcal{Q}^{(n)} \,,$$

$$\mathcal{S}^{ij} J_{ij} \mathcal{Q}^{(n)} := -\left(\mathcal{S}^{(2)} \partial^{(-2)} - n \, \mathcal{S}^{(0)}\right) \mathcal{Q}^{(n)} \,, \qquad \partial^{(-2)} := \frac{1}{(v, u)} u^i \frac{\partial}{\partial v^i} \,, \qquad (1.20)$$

where

$$\xi := \xi^{\mathrm{a}} \mathcal{D}_{\mathrm{a}} + \xi_i^\alpha \mathcal{D}_\alpha^i + \bar{\xi}_{\dot\alpha}^i \bar{\mathcal{D}}_i^{\dot\alpha}$$

is an $\mathcal{N} = 2$ AdS Killing vector field, see section 2 for the definition. In (1.20) we have introduced

$$\mathcal{S}^{(2)} := v_i v_j \mathcal{S}^{ij} \,, \qquad \mathcal{S}^{(0)} := \frac{1}{(v, u)} v_i u_j \mathcal{S}^{ij} \,. \qquad (1.21)$$

The transformation law (1.20) involves an additional isotwistor, u_i, which is only subject to the condition $(v, u) := v^i u_i \neq 0$, and is otherwise completely arbitrary. Both $\mathcal{Q}^{(n)}$ and $\delta_\xi \mathcal{Q}^{(n)}$ are independent of u_i. It is seen that the projective supermultiplets live in the AdS projective superspace $\mathrm{AdS}^{4|8} \times \mathbb{C}P^1$.

In the family of projective multiplets, a generalized conjugation, $\mathcal{Q}^{(n)}(v^i) \to \breve{\mathcal{Q}}^{(n)}(v^i)$, is defined as follows:

$$\breve{\mathcal{Q}}^{(n)}(v) := \bar{\mathcal{Q}}^{(n)}\left(\bar{v} \to \mathrm{i}\,\sigma_2\, v\right) \,, \qquad (1.22)$$

with $\bar{\mathcal{Q}}^{(n)}(\bar{v})$ the complex conjugate of $\mathcal{Q}^{(n)}(v)$ and σ_2 the second Pauli matrix. One can check that $\breve{\mathcal{Q}}^{(n)}(v)$ is a projective multiplet of weight n. One can also see that $\breve{\breve{\mathcal{Q}}}^{(n)} = (-1)^n \mathcal{Q}^{(n)}$, and therefore real supermultiplets can be consistently defined when n is even. The $\breve{\mathcal{Q}}^{(n)}$ is called the smile-conjugate of $\mathcal{Q}^{(n)}$.

To describe the dynamics of supersymmetric field theories in $\mathrm{AdS}^{4|8}$, the following supersymmetric action principle can be used

$$S = \frac{1}{2\pi} \oint_C (v, \mathrm{d}v) \int \mathrm{d}^4 x \, \mathrm{d}^4\theta \, \mathrm{d}^4\bar\theta \, \mathbb{E} \, \frac{\mathcal{L}^{(2)}}{(\mathcal{S}^{(2)})^2} \,, \qquad \mathbb{E}^{-1} = \mathrm{Ber}(E_\mathcal{A}{}^\mathcal{M}) \,, \qquad (1.23)$$

with $(v, \mathrm{d}v) := v^i \mathrm{d}v_i$. Here the Lagrangian is a real weight-two projective multiplet in $\mathrm{AdS}^{4|8}$. The first integral in (1.23) is along a contour in $\mathbb{C}P^1$ parametrized by complex homogeneous coordinates v^i. The second integral is over $\mathrm{AdS}^{4|8}$.

In this paper, we mostly concentrate on studying a certain class of $\mathcal{N} = 2$ supersymmetric σ-models in AdS_4 introduced in [10]. Such a theory is a system of interacting *covariant arctic weight-zero* multiplets

$$\Upsilon^I(v) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I \,, \qquad \zeta := \frac{v^2}{v^1} \qquad (1.24)$$

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and their smile-conjugates

$$\breve{\Upsilon}^{\bar{I}}(v) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \, \bar{\Upsilon}_n^{\bar{I}} \, . \qquad (1.25)$$

described by the Lagrangian

$$\mathcal{L}^{(2)} = \frac{1}{2s} \mathcal{S}^{(2)} \, K(\Upsilon, \breve{\Upsilon}) \, , \qquad (1.26)$$

with $s = \sqrt{\frac{1}{2} \mathcal{S}^{ij} \mathcal{S}_{ij}}$. Here $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ is the Kähler potential of a real analytic Kähler manifold $\mathcal{X}$. The interpretation of K as a Kähler potential is consistent, since the action generated by (1.26) turns out to be invariant under Kähler transformations of the form

$$K(\Upsilon, \breve{\Upsilon}) \;\rightarrow\; K(\Upsilon, \breve{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\breve{\Upsilon}) \, , \qquad (1.27)$$

with $\Lambda(\Phi^I)$ a holomorphic function. The target space $\mathcal{M}$ of this σ-model proves to be an open domain of the zero section of the cotangent bundle of $\mathcal{X}$, $\mathcal{M} \subset T^*\mathcal{X}$. This can be shown by generalizing the flat-superspace considerations of [27, 19].

The $\mathcal{N} = 2$ supersymmetric σ-models defined by (1.9) and (1.23), (1.26) are off-shell. This is a built-in property of the latter theory formulated in the $\mathcal{N} = 2$ AdS superspace. The off-shell nature of the former theory is a non-trivial result established in [1, 2]. Each hypermultiplet in the model (1.9) is described in terms of $8 + 8$ degrees of freedom which are packaged into two $\mathcal{N} = 1$ chiral superfields and their conjugates. On the other hand, each arctic multiplet $\Upsilon^I(v)$, eq. (1.24), contains an infinite number of ordinary fields, most of which are auxiliary. One of the main virtues of the σ-model (1.23), (1.26) is that its Lagrangian (1.26) is given in terms of an arbitrary function $K(\Phi^I, \bar{\Phi}^{\bar{J}})$. Therefore, this σ-model formulation allows us, in principle, to generate hyperkähler manifolds as well as to address the problem of deformations of such manifolds. To achieve these goals, however, we have to develop techniques to eliminate the infinite number of auxiliary fields. In particular, we have to understand how to relate the σ-model (1.23), (1.26) to the $\mathcal{N} = 1$ formulation (1.9).

Due to the analyticity constraints $\mathcal{D}_\alpha^{(1)} \Upsilon^I = \bar{\mathcal{D}}_{\dot{\alpha}}^{(1)} \Upsilon^I = 0$, the Taylor coefficients Υ_n^I in (1.24) are constrained $\mathcal{N} = 2$ superfields. Once restricted to an $\mathcal{N} = 1$ subspace of the $\mathcal{N} = 2$ superspace $\mathrm{AdS}^{4|8}$, the coefficients $\Upsilon_2^I, \Upsilon_3^I, \ldots,$ can be shown to be unconstrained $\mathcal{N} = 1$ superfields. Upon reducing the superspace integral in (1.23) with Lagrangian (1.26) to that over the $\mathcal{N} = 1$ subspace chosen, it can be shown that the superfields $\Upsilon_2^I, \Upsilon_3^I, \ldots,$ appear in the action without derivatives, and therefore they are purely auxiliary and can be eliminated algebraically using their equations of motion. A natural option for how

to define this $\mathcal{N} = 1$ subspace of AdS$^{4|8}$ is to choose the condition (1.3a) and embed AdS$^{4|4}$ into AdS$^{4|8}$ using the procedure described above. However, such a set-up does not allow us to make use of the methods which have been developed for the general $\mathcal{N} = 2$ supersymmetric σ-models in Minkowski space [18]. In other words, some conceptually new techniques are required if AdS$^{4|4}$ is chosen as the desired $\mathcal{N} = 1$ subspace of AdS$^{4|8}$. Such techniques have not yet been developed. On the other hand, the problem of eliminating the auxiliary superfields can be reduced to that studied in [18, 19, 20, 21, 22, 23, 28, 29] if we choose (1.3a) and follow the five-dimensional construction of [16] to foliate AdS$^{4|8}$ into a family of three-dimensional $\mathcal{N} = 2$ Minkowski superspaces.

We will show that the choice (1.3b) leads, upon elimination of the auxiliary superfields in the σ-model defined by eqs. (1.23) and (1.26), to an action in Poincaré coordinates[8]

$$ S = \int \frac{\mathrm{d}z}{(sz)^2} \left\{ \int \mathrm{d}^3 x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}}\, \mathbb{K}(\phi, \bar\phi) + \left(\mathrm{i} \int \mathrm{d}^3 x\, \mathrm{d}^2\boldsymbol{\theta}\, H_{\mathrm{a}}(\phi) \partial_z \phi^{\mathrm{a}} + \mathrm{c.c.} \right) \right\} . \qquad (1.28) $$

The coordinates $(x, \boldsymbol{\theta}, \bar{\boldsymbol{\theta}})$ parametrize the 3D $\mathcal{N} = 2$ Minkowski superspace $\mathbb{R}^{3|4}$ lying at constant values of z. The real function $\mathbb{K}(\phi, \bar\phi)$ is a Kähler potential of the hyperkähler target space $\mathcal{M}$. The three-dimensional $\mathcal{N} = 2$ chiral superfields ϕ^{a} are complex coordinates with respect to the complex structure $\mathbb{J}$ defined by (1.16). Finally, $H = H_{\mathrm{a}}(\phi) \mathrm{d}\phi^{\mathrm{a}}$ is a globally defined holomorphic (1,0) form on $\mathcal{M}$. Several additional geometric requirements are imposed, which we will discuss.

This paper is organized as follows. In section 2 we introduce two invariant tensors of the $\mathcal{N} = 2$ AdS supergroup OSp(2|4): the intrinsic vector multiplet and the intrinsic hypermultiplet. The latter is then used to realize general $\mathcal{N} = 2$ superconformal σ-models as a subclass of the σ-model family (1.26). The main thrust of section 3 is to show how the off-shell supersymmetric σ-models in AdS described by (1.26) can be reformulated in terms of $\mathcal{N} = 1$ chiral superfields in AdS, that is in the form (1.9). We also describe gauged $\mathcal{N} = 2$ supersymmetric σ-models in the AdS projective superspace AdS$^{4|8} \times \mathbb{C}P^1$ and their reformulation in terms of $\mathcal{N} = 1$ chiral superfields on AdS$^{4|4}$. Section 4 provides a new conformally flat realization for AdS$^{4|8}$ with the key property that this superspace becomes foliated into a union of 3D $\mathcal{N} = 4$ flat superspaces with a real central charge (to be called 3D $\mathcal{N} = 4$ central charge superspace) corresponding to a derivative in the fourth dimension. In section 5 we introduce a new set of Grassmann variables for 3D $\mathcal{N} = 4$ central charge superspace which provides the simplest embedding of 3D $\mathcal{N} = 2$ Minkowski superspace without central charge. This technical construction (to be referred

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[8]The definition of the Grassmann coordinates $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ will be given later.

to as the 3D foliated frame) allows us to reformulate general supersymmetric theories in $\text{AdS}^{4|8} \times \mathbb{C}P^1$ in terms of flat projective supermultiplets. In section 6 we repeat most of the analysis of section 3 using the 3D foliated frame. One of the advantages of this frame, as compared to the AdS frame used in section 3, is that we can explicitly construct the hyperkähler potential for a large class of $\mathcal{N} = 2$ supersymmetric σ-models in AdS. Section 7 is devoted to the analysis of the most general $\mathcal{N} = 2$ supersymmetric σ-model in AdS using the 3D foliation. In section 8 we describe the general geometric features of the hyperkähler target spaces of $\mathcal{N} = 2$ supersymmetric σ-models in AdS. Section 9 is concerned with the $\mathcal{N} = 2$ AdS supersymmetric σ-model on $T^*\mathbb{C}P^n$. This is the only nontrivial example of a nonlinear σ-model in AdS in which we have been able to explicitly eliminate the auxiliary superfields in the AdS frame. Our main findings are summarized in section 10. The main body of this paper is accompanied by two technical appendices. Appendix A describes the explicit form of the Killing vector fields of $\text{AdS}^{4|8}$ in the 3D foliated frame. Appendix B describes the tropical prepotential for the intrinsic vector multiplet in the 3D foliated frame.

2 Intrinsic vector multiplet and hypermultiplet

In four-dimensional $\mathcal{N} = 2$ Minkowski superspace, the standard mechanism to make a charged off-shell hypermultiplet massive [30, 31] consists in coupling the hypermultiplet to a frozen $\text{U}(1)$ vector multiplet such that its chiral field strength W is constant. The same procedure also works in five-dimensional $\mathcal{N} = 1$ Poincaré supersymmetry where the field strength of a vector multiplet, W, is real [32]. Applying this mechanism to an off-shell 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ supersymmetric σ-model (for this the target space has to possess a tri-holomorphic isometry) generates a superpotential [32]. In this section we first discuss an AdS analogue of the frozen vector multiplet – the intrinsic vector multiplet [10], which will be used in subsequent sections. We also introduce a covariantly constant hypermultiplet which proves to be closely related to the geometry of $\text{AdS}^{4|8}$. Making use of this hypermultiplet allows us to realize general $\mathcal{N} = 2$ superconformal σ-models as a subclass of the models (1.26).

2.1 Intrinsic vector multiplet

Consider an Abelian vector multiplet in AdS$^{4|8}$. It can be described by gauge-covariant derivatives

$$\boldsymbol{\mathcal{D}}_{\mathcal{A}} = \mathcal{D}_{\mathcal{A}} + \mathrm{i}V_{\mathcal{A}}\hat{e} \ , \tag{2.1}$$

with $V_{\mathcal{A}}$ the gauge one-form, and $\hat{e}$ the generator of the U(1) gauge group. The gauge-covariant derivatives are subject to the anti-commutation relations

$$\{\boldsymbol{\mathcal{D}}_{\alpha}^{i}, \boldsymbol{\mathcal{D}}_{\beta}^{j}\} = 4\mathcal{S}^{ij}M_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}\left(\mathcal{S}^{kl}J_{kl} + \mathrm{i}\bar{\mathcal{W}}\hat{e}\right) \ , \tag{2.2a}$$

$$\{\bar{\boldsymbol{\mathcal{D}}}_{\dot\alpha i}, \bar{\boldsymbol{\mathcal{D}}}_{\dot\beta j}\} = -4\mathcal{S}_{ij}\bar{M}_{\dot\alpha\dot\beta} + 2\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{ij}\left(\mathcal{S}^{kl}J_{kl} + \mathrm{i}\mathcal{W}\hat{e}\right) \ , \tag{2.2b}$$

$$\{\boldsymbol{\mathcal{D}}_{\alpha}^{i}, \bar{\boldsymbol{\mathcal{D}}}_{\dot\beta j}\} = -2\mathrm{i}\delta_{j}^{i}(\sigma^{c})_{\alpha\dot\beta}\boldsymbol{\mathcal{D}}_{c} \tag{2.2c}$$

which are obtained by combining the AdS algebra of covariant derivatives, (1.2), with that describing the U(1) vector multiplet in Minkowski superspace [33]. Here the field strength $\mathcal{W}$ is covariantly chiral,

$$\bar{\mathcal{D}}_{i}^{\dot\alpha}\mathcal{W} = 0 \ , \tag{2.3}$$

and obeys the Bianchi identity

$$\left(\mathcal{D}^{\alpha(i}\mathcal{D}_{\alpha}^{j)} + 4\mathcal{S}^{ij}\right)\mathcal{W} = \left(\bar{\mathcal{D}}_{\dot\alpha}^{(i}\bar{\mathcal{D}}^{j)\dot\alpha} + 4\mathcal{S}^{ij}\right)\bar{\mathcal{W}} \ . \tag{2.4}$$

Following [10], a U(1) vector multiplet in AdS$^{4|8}$ is called intrinsic if its field strength is constant,

$$\mathcal{W} = 1 \ . \tag{2.5}$$

This condition is consistent with the Bianchi identity (2.4). Such a vector multiplet is 'frozen' in the sense that it has no propagating degrees of freedom. As will be shown in the next section, it is completely determined by the geometry of AdS$^{4|8}$.

2.2 Intrinsic hypermultiplet

The Fayet-Sohnius formulation for the hypermultiplet [34, 35] can be extended to the case of AdS [2, 36]. A charged off-shell hypermultiplet in AdS is described by a two-component superfield[9] q_i and its conjugate $\bar{q}^i := \overline{q_i}$ (such that $\overline{q^i} = -\bar{q}_i$) subject to the

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[9]Isospinor indices are raised and lowed using antisymmetric tensors ε^{ij} and ε_{ij} normalized by $\varepsilon^{\underline{12}} = \varepsilon_{\underline{21}} = 1$. The rules are: $q^i = \varepsilon^{ij}q_j$ and $q_i = \varepsilon_{ij}q^j$.

constraints

$$\mathcal{D}_\alpha^{(i} q^{j)} = \bar{\mathcal{D}}_{\dot\alpha}^{(i} q^{j)} = 0 \ . \tag{2.6}$$

The action of $\mathcal{S}^{kl} J_{kl}$ on q_i is not assumed to be fixed at our will. Instead it is determined by the constraints to be

$$\mathcal{S}^{kl} J_{kl} = \Delta + \boldsymbol{J} \ , \qquad \boldsymbol{J} q_i := -\mathcal{S}_i{}^j q_j \ , \tag{2.7}$$

where Δ takes on the role of a central charge as it commutes with the covariant derivatives,

$$[\Delta, \mathcal{D}_\alpha^i] = [\Delta, \bar{\mathcal{D}}_{\dot\alpha i}] = 0 \ . \tag{2.8}$$

Setting $\Delta q_i = 0$ is equivalent to the equation of motion for a massless hypermultiplet.

The covariantly constant torsion tensor of AdS$^{4|8}$, $\mathcal{S}_{ij}$, can always be represented in the form

$$\mathcal{S}_{ij} = 2\mathrm{i}\, q_{(i} \bar{q}_{j)} \ , \qquad \bar{q}^i := \overline{q_i} \ , \tag{2.9}$$

for some isospinor q_i defined modulo arbitrary phase transformations $q_i \to \mathrm{e}^{\mathrm{i}\varphi} q_i$, with $\bar\varphi = \varphi$. Introducing $|q|^2 := \bar{q}^i q_i$, we easily obtain

$$\mathcal{S}_i{}^j q_j = \mathrm{i}|q|^2 q_i \ , \qquad \mathcal{S}^i{}_j \bar{q}^j = \mathrm{i}|q|^2 \bar{q}^i \ , \tag{2.10}$$

as well as

$$s \equiv \sqrt{\tfrac{1}{2} \mathcal{S}^{ij} \mathcal{S}_{ij}} = |q|^2 \ . \tag{2.11}$$

This shows that $|q|$ is constant. The freedom in the definition of q_i can be fixed by requiring it to be gauge-covariantly constant,

$$\mathcal{D}_{\mathcal{A}} q_i = 0 \ , \qquad \Delta q_i = 0 \ , \tag{2.12}$$

where the derivatives $\mathcal{D}_{\mathcal{A}}$ correspond to the intrinsic vector multiplet. In accordance with (2.2), (2.10) and (2.11), the integrability condition for this constraint is

$$\hat{e} q_i = s q_i \ . \tag{2.13}$$

This frozen hypermultiplet will be called *intrinsic*.

The isometry group of AdS$^{4|8}$, OSp(2|4), is generated by the corresponding Killing vector fields. A real vector field in AdS$^{4|4}$ corresponding to the first-order operator

$$\xi := \xi^{\mathcal{A}} \mathcal{D}_{\mathcal{A}} = \xi^a \mathcal{D}_a + \xi_i^\alpha \mathcal{D}_\alpha^i + \bar\xi_{\dot\alpha}^i \bar{\mathcal{D}}_i^{\dot\alpha} \tag{2.14}$$

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is said to be a Killing vector field if it obeys the master equation

$$[\xi + \frac{1}{2}\lambda^{cd}M_{cd} + 2\varepsilon\mathcal{J}, \mathcal{D}_\alpha^i] = 0 \ , \qquad \mathcal{J} := \mathcal{S}^{kl}J_{kl} \ , \tag{2.15}$$

for uniquely determined parameters λ^{cd} and ε generating Lorentz and U(1) transformations respectively. The explicit expressions for these parameters are

$$\lambda_{ab} = \mathcal{D}_{[a}\xi_{b]} \ , \qquad \varepsilon = \frac{1}{8}\mathcal{S}^{ij}\mathcal{D}_{\alpha i}\xi_j^\alpha \ , \tag{2.16}$$

see [10] for a derivation. If U is a tensor superfield on $\mathcal{N} = 2$ AdS superspace, its infinitesimal transformation associated with ξ is

$$\delta_{\mathrm{AdS}}U = -\xi U - \frac{1}{2}\lambda^{cd}M_{cd}U - 2\varepsilon\mathcal{J}U \ . \tag{2.17}$$

The torsion of $\mathcal{N} = 2$ AdS superspace, $\mathcal{S}^{ij}$, is an invariant tensor, $\delta_{\mathrm{AdS}}\mathcal{S}^{ij} = 0$. Combining δ_{AdS} with a certain U(1) gauge transformation,

$$\hat{\delta}_{\mathrm{AdS}} := \delta_{\mathrm{AdS}} - 2\mathrm{i}\varepsilon\hat{e} \ , \tag{2.18}$$

we can see that the intrinsic hypermultiplet, q_i, is invariant,

$$\hat{\delta}_{\mathrm{AdS}}q_i = 0 \ . \tag{2.19}$$

The defining property of $\hat{\delta}_{\mathrm{AdS}}$ is that the gauge-covariant derivatives do not change,

$$[\xi^\mathcal{C}\mathcal{D}_\mathcal{C} + \frac{1}{2}\lambda^{cd}M_{cd} + 2\varepsilon\mathcal{J} + 2\mathrm{i}\varepsilon\hat{e}, \mathcal{D}_A] = 0 \ , \tag{2.20}$$

where we have used the identity $\xi_\alpha^i = \mathcal{D}_\alpha^i\varepsilon$ derived in [10].

2.3   Maximally symmetric solution for $\mathcal{N} = 2$ AdS supergravity

The intrinsic vector multiplet and the intrinsic hypermultiplet naturally originate in the context of a maximally supersymmetric solution for $\mathcal{N} = 2$ supergravity with a cosmological term if one uses the off-shell supergravity formulation of [37] with the following compensators: the vector multiplet and the hypermultiplet with an intrinsic central charge (see [38] for an early list of off-shell formulations for $\mathcal{N} = 2$ supergravity). The supergravity equations of motion can be shown to be (see [17, 39] for a derivation)

$$\frac{1}{\kappa^2}\Sigma^{ij} = \mathrm{i}e\bar{q}^{(i}q^{j)} \ , \tag{2.21a}$$

$$\frac{1}{\kappa^2}\mathcal{W}\bar{\mathcal{W}} = \frac{1}{2}|q|^2 \ , \tag{2.21b}$$

$$\Delta q_i = 0 \ , \tag{2.21c}$$

where κ is the gravitational constant, e denotes the U(1) charge of q_i related to the cosmological constant, and

$$\Sigma^{ij} = \frac{1}{4}\Big(\mathcal{D}^{\alpha(i}\mathcal{D}^{j)}_\alpha + 4\mathcal{S}^{ij}\Big)\mathcal{W} = \frac{1}{4}\Big(\bar{\mathcal{D}}^{(i}_{\dot\alpha}\bar{\mathcal{D}}^{j)\dot\alpha} + 4\bar{\mathcal{S}}^{ij}\Big)\bar{\mathcal{W}} \ . \tag{2.22}$$

We assume here that the Weyl multiplet is described using the superspace formulation for $\mathcal{N} = 2$ conformal supergravity given in [11] (in this formulation, the torsion $\mathcal{S}^{ij}$ is complex). The relations (2.21a) and (2.21c) are the equations of motion for the vector compensator and the hypermultiplet respectively, while eq. (2.21b) corresponds to the gravitational superfield (see [39] for more details). The equations (2.21) can be shown to be super-Weyl invariant (see also subsection 4.2).

We are interested in a supergravity solution with vanishing super-Weyl tensor, that is $\mathcal{W}_{\alpha\beta} = 0$. The super-Weyl invariance can be used to choose the gauge $\mathcal{W} = 1$, in which $\mathcal{S}^{ij}$ becomes real, as a consequence of (2.22). Then, the relations (2.21a), (2.21b) and (2.22) lead to

$$\mathcal{S}^{ij} = \frac{2\mathrm{i}e}{|q|^2}q^{(i}\bar{q}^{j)} \ , \tag{2.23}$$

which implies the consistency condition $e = s$. The hypermultiplet constraints $\mathcal{D}^{(i}_\alpha q^{j)} = \bar{\mathcal{D}}^{(i}_{\dot\alpha}q^{j)} = 0$ and the supergravity equation of motion (2.21b), $|q|^2 = 2/\kappa^2 = \text{const}$, imply that q_i is covariantly constant, $\mathcal{D}_A q_i = 0$. As a result, it can be seen that the algebra of the supergravity covariant derivatives [11] reduces to (2.2), and therefore the supergravity solution constructed describes the AdS geometry. To completely reproduce the construction of subsection 2.2, it only remains to normalize $|q|^2 = e$.

Two comments are in order. Firstly, there exists an off-shell formulation for $\mathcal{N} = 2$ supergravity [11] in which the hypermultiplet compensator is described in terms of a co-variant weight-one arctic multiplet $q^{(1)}(v)$ and its conjugate $\breve{q}^{(1)}(v)$, which are coupled to the vector compensator in the case of a non-zero cosmological constant. In this formulation, the hypermultiplet has no central charge, $\Delta q^{(1)} \equiv 0$ off the mass shell. The hypermultiplet equation of motion is $q^{(1)}(v) = q_i v^i$. The equations (2.21a) and (2.21b) remain the same. Secondly, one can describe $\mathcal{N} = 2$ AdS supergravity using the off-shell formulation of [38] which makes use of two compensators: the vector multiplet and the tensor multiplet. The superspace description of the AdS solution within this supergravity formulation is given in [17].

2.4   Off-shell $\mathcal{N} = 2$ superconformal σ-models

Consider a system of interacting *covariant weight-zero arctic* multiplets $\Upsilon^I(v)$ and their smile-conjugates $\breve{\Upsilon}^{\bar{I}}(v)$ described by the Lagrangian

$$\mathcal{L}^{(2)} = \frac{1}{2s}\mathcal{S}^{(2)} K(\Upsilon, \breve{\Upsilon}) . \tag{2.24}$$

Here $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ is the Kähler potential of a Kähler cone; it obeys the homogeneity condition

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \tag{2.25}$$

With the homogeneity condition imposed, no Kähler invariance survives. The Kähler cone is the target space of an $\mathcal{N} = 1$ superconformal σ-model, see e.g. [28]. We can associate with $\Upsilon(v)$ a weight-one arctic multiplet $\mathbf{\Upsilon}^{(1)}(v)$ and its smile-conjugate $\breve{\mathbf{\Upsilon}}^{(1)}(v)$ defined by

$$\mathbf{\Upsilon}^{(1)}(v) := \frac{1}{\sqrt{s}}q^{(1)}(v)\Upsilon(v) , \qquad \breve{\mathbf{\Upsilon}}^{(1)}(v) := \frac{1}{\sqrt{s}}\breve{q}^{(1)}(v)\breve{\Upsilon}(v) , \tag{2.26}$$

where we have defined $q^{(1)} := q_i v^i$ and $\breve{q}^{(1)} := \bar{q}_i v^i$, with q_i being the intrinsic hypermultiplet. In terms of the weight-one projective superfields $\mathbf{\Upsilon}^{(1)}(v)$ and $\breve{\mathbf{\Upsilon}}^{(1)}(v)$, the Lagrangian (2.24) takes the form

$$\mathcal{L}^{(2)} = \mathrm{i}\, K(\mathbf{\Upsilon}^{(1)}, \breve{\mathbf{\Upsilon}}^{(1)}) . \tag{2.27}$$

where we have used the relation (2.9). The σ-model obtained is $\mathcal{N} = 2$ superconformal. Its Lagrangian (2.27) has the same form an in the super-Poincaré case [40].

3 σ-models from projective superspace: AdS frame

In the introduction, we briefly reviewed the results of [1, 2] regarding the general form of $\mathcal{N} = 2$ supersymmetric σ-models written in terms of the $\mathcal{N} = 1$ AdS superspace $\mathrm{AdS}^{4|4}$. In this section, we will demonstrate explicitly how these models come about from a projective superspace context.

The starting point is the general projective superspace action in $\mathrm{AdS}^{4|8}$, eq. (1.23). Let us make the choice (1.3a). Using the techniques described in [10], this action can be rewritten in $\mathrm{AdS}^{4|4}$ as

$$S = \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \mathcal{L} \tag{3.1}$$

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where the $\mathcal{N} = 1$ AdS Lagrangian $\mathcal{L}$ is given by a contour integral

$$\mathcal{L} = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\,\mathcal{L}^{[2]}\Big|\ ,\qquad \mathcal{L}^{[2]}(\zeta) := \frac{1}{\mathrm{i}(v^{\underline{1}})^2\zeta}\mathcal{L}^{(2)}(v)\ , \tag{3.2}$$

with the bar-projection defined by (1.7). Specializing to the σ-model described by (1.26), we find

$$\mathcal{L} = \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\,\mathcal{S}^{[2]}K(\Upsilon,\breve{\Upsilon})\Big|\ ,\qquad \mathcal{S}^{[2]}(\zeta) := \frac{1}{\mathrm{i}(v^{\underline{1}})^2\zeta}\mathcal{S}^{(2)}(v) = \frac{\mathrm{i}\mu}{\zeta} + \mathrm{i}\bar{\mu}\zeta\ . \tag{3.3}$$

In this context, $s = |\mu|$. In what follows, the bar-projection is not indicated explicitly.

We will analyze this σ-model action in several stages. First, we will directly evaluate it without gauging any of the isometries which the Kähler space with the Kähler potential K might possess. This leads to the class of actions discussed in [1, 2]. Then we will consider the case where the Kähler space possesses a holomorphic isometry, and demonstrate how this leads to a tri-holomorphic isometry of the hyperkähler target space, in terms of the σ-model formulated using $\mathcal{N} = 1$ chiral superfields. When one such isometry is gauged by the intrinsic vector multiplet of AdS a superpotential naturally emerges.[10]

3.1   The ungauged case

Upon projection to $\mathcal{N} = 1$ AdS superspace, which requires the standard choice (1.3a), the weight-zero arctic multiplet Υ^I consists of an infinite set of $\mathcal{N} = 1$ superfields, (1.24). Using the analyticity constraints (1.18), one can show that the lowest two components,

$$\Phi^I := \Upsilon_0^I\ ,\qquad \Sigma^I := \Upsilon_1^I \tag{3.4}$$

are constrained $\mathcal{N} = 1$ superfields: Φ^I is covariantly chiral and Σ^I is covariantly complex linear,

$$\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^I = 0\ ,\qquad (\bar{\mathcal{D}}^2 - 4\mu)\Sigma^I = 0\ . \tag{3.5}$$

All the other components of Υ^I are unconstrained complex $\mathcal{N} = 1$ superfields. These superfields appear in the action without derivatives, and therefore they are auxiliary. The superfields Φ^I and Σ^I are physical.

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[10]Due to the properties of $\mathcal{N} = 1$ AdS, the seemingly more general case involving a gauged isometry is also contained within the class of actions considered in [1, 2], even though tri-holomorphic isometries were not considered explicitly in that work.

For our subsequent analysis, it is important to work out the supersymmetry transformation laws of the physical superfields Φ^I and Σ^I. We begin by recalling that the weight-zero arctic multiplet transforms under the full $\mathcal{N} = 2$ AdS supergroup, $OSp(2|4)$, according to eq. (1.20). Upon $\mathcal{N} = 1$ projection, this $\mathcal{N} = 2$ transformation decomposes into two different transformations in $\mathcal{N} = 1$ AdS superspace [2, 10] which are:

(i) An $\mathcal{N} = 1$ AdS isometry transformation

$$\delta \Upsilon^I(\zeta) = -\xi \Upsilon^I(\zeta) \ , \qquad \xi := \xi^a \mathcal{D}_a + \xi^\alpha \mathcal{D}_\alpha + \bar{\xi}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \tag{3.6}$$

generated by an arbitrary $\mathcal{N} = 1$ Killing vector field $\xi^A = (\xi^a, \xi^\alpha, \bar{\xi}_{\dot{\alpha}})$. Such a vector field obeys the Killing equation

$$[\xi + \frac{1}{2}\lambda^{bc} M_{bc}, \mathcal{D}_A] \equiv [\xi + \lambda^{\beta\gamma} M_{\beta\gamma} + \bar{\lambda}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}, \mathcal{D}_A] = 0 \ , \qquad \lambda_{\alpha\beta} = \mathcal{D}_{(\alpha}\lambda_{\beta)} \ . \tag{3.7}$$

These Killing vector fields generate the isometry group of $AdS^{4|4}$, $OSp(1|4)$.

(ii) An extended supersymmetry transformation

$$\delta \Upsilon^I(\zeta) = -(\zeta \varepsilon^\alpha \mathcal{D}_\alpha - \frac{1}{\zeta} \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} - 2\varepsilon \mu \partial_\zeta - 2\varepsilon \zeta^2 \bar{\mu} \partial_\zeta) \Upsilon^I(\zeta) \ , \qquad \varepsilon_\alpha := \mathcal{D}_\alpha \varepsilon \ , \tag{3.8}$$

where ε is a real superfield obeying the constraints [41]

$$(\bar{\mathcal{D}}^2 - 4\mu)\varepsilon = (\mathcal{D}^2 - 4\bar{\mu})\varepsilon = 0 \ , \qquad \bar{\mathcal{D}}_{\dot{\beta}} \mathcal{D}_\alpha \varepsilon = \mathcal{D}_\alpha \bar{\mathcal{D}}_{\dot{\beta}} \varepsilon = 0 \ . \tag{3.9}$$

The first constraint implies that ε is real linear. The superfield parameter ε contains two components: (i) a bosonic parameter ρ which is defined by $\varepsilon|_{\theta=0} = \rho|\mu|^{-1}$ and describes the $O(2)$ rotations; and (ii) a fermionic parameter $\epsilon_\alpha := \mathcal{D}_\alpha \varepsilon|_{\theta=0}$ along with its conjugate, which generate the second supersymmetry.

It follows from (3.8) that the extended supersymmetry transformation acts on the lowest two component fields as

$$\delta \Phi^I = \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \Sigma^I + 2\varepsilon \mu \Sigma^I \tag{3.10a}$$

$$= \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon \Sigma^I) \tag{3.10b}$$

and

$$\delta \Sigma^I = -\varepsilon^\alpha \mathcal{D}_\alpha \Phi^I + \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \Upsilon_2^I + 4\mu \varepsilon \Upsilon_2^I = -\varepsilon^\alpha \mathcal{D}_\alpha \Phi^I + \bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\varepsilon}^{\dot{\alpha}} \Upsilon_2^I) \ . \tag{3.11}$$

We have written the transformation law of Φ^I in two ways by exploiting the complex linearity of Σ^I; in (3.10b) the transformation is manifestly chiral. Similarly, we have written (3.11) to emphasize that $\delta \Sigma^I$ is complex linear.

The $\mathcal{N} = 1$ Lagrangian $\mathcal{L}(\Upsilon_n, \bar{\Upsilon}_n)$, eq. (3.3), is a function of not only Φ^I and Σ^I but also the infinite set of unconstrained $\mathcal{N} = 1$ superfields Υ_2^I, $\Upsilon_3^I, \ldots$ Because these superfields are unconstrained, they can be eliminated (at least in principle) using their algebraic equations of motion,

$$0 = \frac{\partial \mathcal{L}}{\partial \Upsilon_n^I} = \frac{1}{2s} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i} \zeta} \mathcal{S}^{[2]} \frac{\partial K}{\partial \Upsilon^I} \zeta^n \,, \qquad n \geq 2 \,; \tag{3.12a}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\Upsilon}_n^{\bar{J}}} = \frac{1}{2s} \oint_C \frac{\mathrm{d}\zeta}{2\pi \mathrm{i} \zeta} \mathcal{S}^{[2]} \frac{\partial K}{\partial \breve{\bar{\Upsilon}}^{\bar{J}}} (-\zeta)^{-n} \,, \qquad n \geq 2 \,. \tag{3.12b}$$

This step is the nontrivial technical challenge of the projective superspace approach: we must solve an infinite set of algebraic equations. This issue was resolved in Minkowski superspace for a large class of σ-models on cotangent bundles of Hermitian symmetric spaces [18, 19, 20, 21, 22, 23] (see [24] for a review). For these cases, the equations which must be solved in a flat background correspond to the choice $\mathcal{S}^{[2]} = 2s$ – in other words, the choice (1.3b) discussed in the introduction. For the case of interest to us, $\mathcal{S}^{[2]} = \mathrm{i}\mu/\zeta + \mathrm{i}\bar{\mu}\zeta$, the problem remains unsolved except for the special case described in section 9. Nevertheless, we may proceed formally by generalizing the flat-superspace analysis of [28, 29].

The σ-model action is manifestly invariant under the $\mathcal{N} = 1$ AdS transformation (3.6), both before and after elimination of the auxiliary superfields. If we assume that the auxiliaries have been eliminated, with $\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma})$ being the resulting Lagrangian, the action must be invariant under the transformations (3.10a) and (3.11) where Υ_2^I is now understood as a function of Φ^I, Σ^I, and their complex conjugates. This follows from the contour integral definition of $\mathcal{L}$. It will be important later that the proof of invariance of the action does *not* require that Σ^I be complex linear, provided one uses the form (3.10a) for the transformation $\delta\Phi^I$.

Alternatively, one can look for a Lagrangian $\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma})$ such that the corresponding action is invariant under extended supersymmetry transformations (3.10) and (3.11), with Υ_2^I some unknown function of Φ^I, Σ^I, and their complex conjugates. One can then check that invariance of the action implies the existence of a complex function Ξ obeying the

equations,[11]

$$\frac{\partial \mathcal{L}}{\partial \Phi^I} + \frac{\partial \mathcal{L}}{\partial \Sigma^J}\frac{\partial \Upsilon_2^J}{\partial \Sigma^I} = \frac{\partial \Xi}{\partial \Sigma^I} \ , \tag{3.13a}$$

$$-\frac{\partial \mathcal{L}}{\partial \bar{\Sigma}^I} + \frac{\partial \mathcal{L}}{\partial \Sigma^J}\frac{\partial \Upsilon_2^J}{\partial \bar{\Phi}^I} = \frac{\partial \Xi}{\partial \bar{\Phi}^I} \ , \tag{3.13b}$$

$$\frac{\partial \mathcal{L}}{\partial \Sigma^J}\frac{\partial \Upsilon_2^J}{\partial \bar{\Sigma}^I} = \frac{\partial \Xi}{\partial \bar{\Sigma}^I} \ . \tag{3.13c}$$

From the contour integral definition of $\mathcal{L}$, one can show that Ξ is given by

$$\Xi := \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\frac{\mathcal{S}^{[2]}}{\zeta}K \ . \tag{3.14}$$

In the case of Minkowski space [29], the equations (3.13) guarantee invariance of the action under the second supersymmetry transformation.[12] In the case of AdS space, however, one finds an additional requirement

$$\mu\Xi - \frac{1}{2}\mu\Sigma^I\frac{\partial \mathcal{L}}{\partial \Phi^I} - \mu\Upsilon_2^I\frac{\partial \mathcal{L}}{\partial \Sigma^I} + \text{c.c.} = \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \tag{3.15}$$

for some chiral superfield $\Lambda(\Phi)$. Making use of the contour integral definitions of Ξ and $\mathcal{L}$, we actually find that a stronger version of this condition is satisfied,

$$\mu\Xi - \frac{1}{2}\mu\Sigma^I\frac{\partial \mathcal{L}}{\partial \Phi^I} - \mu\Upsilon_2^I\frac{\partial \mathcal{L}}{\partial \Sigma^I} + \text{c.c.} = 0 \ . \tag{3.16}$$

This condition (in either form) should be equivalent to the existence of a Killing vector V^μ in the dual geometry which acts as a rotation on the complex structures.

The next step is to perform a duality, exchanging the complex linear superfield Σ^I for a chiral superfield Ψ_I. One introduces the so-called "first-order" action

$$S_{\text{F.O.}} = \int \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, E\left(\mathcal{L} + \Sigma^I\Psi_I + \bar{\Sigma}^{\bar{J}}\bar{\Psi}_{\bar{J}}\right) \ , \tag{3.17}$$

where complex linearity of Σ^I is enforced by the Ψ_I equation of motion. Instead if we apply the equation of motion for the unconstrained Σ^I, we find

$$S_{\text{dual}} = \int \mathrm{d}^4 x\, \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\, E\, \mathcal{K}(\Phi^I, \Psi_I, \bar{\Phi}^{\bar{J}}, \bar{\Psi}_{\bar{J}}) \ , \tag{3.18}$$

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[11]It was argued for the first time in [28, 29] that such a function Ξ must exist in flat backgrounds. We find the same condition in AdS.

[12]The contour integral representation for Ξ is different in flat space, see eq. (6.20).

where $\mathcal{K}$ is defined as

$$\mathcal{K} := \mathcal{L} + \Sigma^I \Psi_I + \bar{\Sigma}^{\bar{J}} \bar{\Psi}_{\bar{J}} \tag{3.19}$$

and Σ^I is chosen to obey

$$\frac{\partial \mathcal{L}}{\partial \Sigma^I} = -\Psi_I . \tag{3.20}$$

We must construct the transformation law of Ψ_I for the dual action. This is easiest to do by first constructing it for the first-order action. We begin by postulating the transformation law of Φ^I. To maintain its chirality, we must choose

$$\delta \Phi^I := \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon \Sigma^I) = \bar{\varepsilon}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \Sigma^I + 2\varepsilon \mu \Sigma^I + \frac{1}{2}\varepsilon(\bar{\mathcal{D}}^2 - 4\mu)\Sigma^I , \tag{3.21}$$

where the last term no longer vanishes. Keeping the same transformation law (3.11) for Σ^I, we find that

$$\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \delta\mathcal{L} = \frac{1}{2} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \Sigma^I (\bar{\mathcal{D}}^2 - 4\mu) \left(\varepsilon \frac{\partial \mathcal{L}}{\partial \Phi^I} \right) + \text{c.c.} \tag{3.22}$$

It follows that we must choose

$$\delta \Psi_I = -\frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\left(\varepsilon \frac{\partial \mathcal{L}}{\partial \Phi^I} \right) . \tag{3.23}$$

The dual action (3.18) is then invariant under

$$\delta \Phi^I = \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\left(\varepsilon \frac{\partial \mathcal{K}}{\partial \Psi_I} \right) , \qquad \delta \Psi_I = -\frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)\left(\varepsilon \frac{\partial \mathcal{K}}{\partial \Phi^I} \right) . \tag{3.24}$$

Now, we have to recall the structure of the extended supersymmetry transformation in the model (1.9). In accordance with [1, 2], let φ^a denote the general chiral coordinate of the Kähler potential $\mathcal{K}$,

$$\delta \varphi^a = \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon \omega^{ab} \mathcal{K}_b) , \tag{3.25}$$

where $(-\omega^{ab})$ is the inverse of the covariantly constant $(2,0)$ holomorphic form ω_{ab} in terms of which two other complex structures J_1 and J_2 of the hyperkähler target space are constructed, eq. (1.11). In our case, the set of chiral superfields is $\varphi^a = (\Phi^I, \Psi_I)$. It follows from (3.24) that

$$\omega^{ab} = \begin{pmatrix} 0 & \delta^I{}_J \\ -\delta_I{}^J & 0 \end{pmatrix} , \qquad \omega_{ab} = \begin{pmatrix} 0 & \delta_I{}^J \\ -\delta^I{}_J & 0 \end{pmatrix} . \tag{3.26}$$

In other words, the coordinates Φ^I and Ψ_I we have found from projective superspace are holomorphic Darboux coordinates for the target space.

3.2 Superpotentials and tri-holomorphic isometries

An obvious question to ask is whether, given an $\mathcal{N} = 2$ supersymmetric σ-model (1.9) invariant under the second supersymmetry transformation (3.25), it is possible to deform the action to include a superpotential,

$$\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \mathcal{K}(\varphi, \bar{\varphi}) + \left(\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathcal{E} \, W(\varphi) + \text{c.c.} \right) , \tag{3.27}$$

with $\mathcal{E}$ the chiral density. However, there is no way to distort the supersymmetry transformation rule (3.25) modulo a trivial symmetry transformation [1, 2]. It follows that the superpotential terms in (3.27) must be separately invariant. In order for this to occur, the integral

$$\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \varepsilon \, (W_a \omega^{ab} \mathcal{K}_b + \text{c.c.}) \tag{3.28}$$

must vanish. Since ε is real linear, we find

$$X^a \mathcal{K}_a + \bar{X}^{\bar{a}} \mathcal{K}_{\bar{a}} = \Lambda(\phi) + \bar{\Lambda}(\bar{\phi}) \tag{3.29}$$

where

$$X^a := -\omega^{ab} W_b \tag{3.30}$$

is a holomorphic Killing vector. Note that $X^a W_a = 0$ by construction. Moreover, from its construction, one can further show that X^a must be tri-holomorphic,

$$\mathcal{L}_X \omega_{ab} = 0 \quad \Longrightarrow \quad \mathcal{L}_X J_A = 0 . \tag{3.31}$$

If we impose the additional requirement that the action (1.9) itself be invariant under the holomorphic isometry X^a, we find the stricter condition

$$\mathcal{L}_X \mathcal{K} = X^a \mathcal{K}_a + \bar{X}^{\bar{a}} \mathcal{K}_{\bar{a}} = 0 . \tag{3.32}$$

Note that the superpotential is automatically invariant, $X^a W_a = 0$. It is straightforward to check that

$$\mathcal{L}_X \mathcal{K} = 0 \quad \Longleftrightarrow \quad \mathcal{L}_V \left(\frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}} \right) = 0 \quad \Longrightarrow \quad [X, V] = 0 . \tag{3.33}$$

There is an interesting geometric interpretation of this construction. We observe that within $\mathcal{N} = 1$ AdS, a superpotential is not distinct from a purely holomorphic contribution to $\mathcal{K}$,

$$\int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathcal{E} \, W + \text{c.c.} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \left(\frac{W}{\mu} + \text{c.c.} \right) . \tag{3.34}$$

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So the addition of a superpotential corresponds to a modification of the Lagrangian $\mathcal{K}$ by

$$\mathcal{K} \to \mathcal{K}' = \mathcal{K} + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}} \; . \tag{3.35}$$

Recall that the target space geometry possesses a U(1) Killing vector V^a which rotates the complex structures. It is given by eq. (1.13). If we modify the original action to include a superpotential, it is natural to absorb this superpotential back into $\mathcal{K}$. This induces the transformation

$$V^a \to V'^a = V^a - \frac{1}{2|\mu|} X^a \; . \tag{3.36}$$

Since both V^a and V'^a should rotate the complex structures in the same way, it follows that X^a must leave them invariant. In other words, X^a must be tri-holomorphic.

This structure emerges naturally in projective superspace if we consider the gauging of holomorphic symmetries of the original σ-model action in projective superspace.

3.2.1   Tri-holomorphic isometries

We now return to the off-shell $\mathcal{N} = 2$ supersymmetric σ-model (3.3). The reader should keep in mind that this theory is associated with some real analytic Kähler manifold $\mathcal{X}$, and $K(\Phi, \bar{\Phi})$ is its Kähler potential in local complex coordinates Φ^I. Suppose $\mathcal{X}$ has a U(1) isometry group generated by a holomorphic Killing vector field

$$X = X^I(\Phi)\partial_I + \bar{X}^{\bar{I}}(\bar{\Phi})\partial_{\bar{I}} \; . \tag{3.37}$$

Under an infinitesimal isometry transformation

$$\mathrm{i}\hat{e}\Phi^I \equiv X\Phi^I = X^I(\Phi) \; , \tag{3.38}$$

the Kähler potential changes as

$$\mathrm{i}\hat{e}K(\Phi, \bar{\Phi}) = X^I\partial_I K + \bar{X}^{\bar{I}}\partial_{\bar{I}}K = F(\Phi) + \bar{F}(\bar{\Phi}) \; , \tag{3.39}$$

for some holomorphic function $F(\Phi)$. Here we have formally introduced the U(1) generator, $\hat{e}$, in order to make contact with the description in terms of the gauge-covariant derivatives (2.1).

The isometry transformation can be extended to the arctic variables of the σ-model (3.3) by analytic continuation:

$$\mathrm{i}\hat{e}\Upsilon^I = X^I(\Upsilon) \; , \tag{3.40}$$

which implies the following transformation of the antarctic variables

$$i\hat{e}\check{\Upsilon}^{\bar{I}} = \bar{X}^{\bar{I}}(\check{\Upsilon}) . \tag{3.41}$$

It immediately follows from (3.40) that the tangent bundle variables (Φ^I, Σ^I) transform as

$$i\hat{e}\Phi^I = X^I(\Phi) , \qquad i\hat{e}\Sigma^I = \Sigma^J\partial_J X^I(\Phi) . \tag{3.42}$$

Due to (3.39), it holds that

$$i\hat{e}K(\Upsilon, \check{\Upsilon}) = X^I\partial_I K + \check{X}^{\bar{J}}\partial_{\bar{J}}K = F(\Upsilon) + \bar{F}(\check{\Upsilon}) . \tag{3.43}$$

Because $K(\Upsilon, \check{\Upsilon})$ is not invariant in the case $F \neq 0$, it turns out that the tangent bundle Lagrangian $\mathcal{L}$ is not invariant either. Observe that

$$i\hat{e}\mathcal{L} = \frac{1}{2s} \oint_C \frac{\mathrm{d}\zeta}{2\pi i\zeta} \mathcal{S}^{[2]} \left(F(\Upsilon) + \bar{F}(\check{\Upsilon}) \right) = \frac{1}{2s} \oint_C \frac{\mathrm{d}\zeta}{2\pi i\zeta} \left(\frac{i\mu}{\zeta} F(\Upsilon) + i\bar{\mu}\zeta \bar{F}(\check{\Upsilon}) \right) , \tag{3.44}$$

where we have dropped terms which do not contribute under the contour integral. Assuming that the contour is around the origin in the ζ-plane, this gives

$$i\hat{e}\mathcal{L} = \frac{i\mu}{2|\mu|}\Sigma^I F_I(\Phi) - \frac{i\bar{\mu}}{2|\mu|}\Sigma^{\bar{J}}\bar{F}_{\bar{J}}(\bar{\Phi}) , \tag{3.45}$$

where $F_I = \partial_I F$. So long as Σ^I is complex linear, this remains a symmetry of the action (3.1). However, when we go to the first order form, eq. (3.17), we must modify the transformation law of Ψ_I:

$$i\hat{e}\Psi_I = -\partial_I X^J\Psi_J - \frac{i\mu}{2|\mu|}F_I . \tag{3.46}$$

The presence of the μ-dependent term implies that Ψ_I does not transform as a holomorphic one-form at the point $(\Phi, \bar{\Phi})$ of $\mathcal{X}$, unlike in the super-Poincaré case [32].

We end up with a holomorphic vector field

$$X = X^a(\varphi)\partial_a + \bar{X}^{\bar{a}}(\bar{\varphi})\partial_{\bar{a}} \tag{3.47}$$

on the hyperkähler target space parametrized by local complex coordinates $\varphi^a = (\Phi^I, \Psi_I)$. This vector field acts on the complex coordinates as follows:

$$i\hat{e}\Phi^I \equiv X\Phi^I = X^I(\Phi) , \qquad i\hat{e}\Psi_I \equiv X\Psi_I = -\partial_I X^J(\Phi)\Psi_J - \frac{i\mu}{2|\mu|}F_I(\Phi) . \tag{3.48}$$

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The Kähler potential $\mathcal{K}$, which serves as the $\mathcal{N} = 1$ AdS Lagrangian for the cotangent bundle (3.18), remains invariant,

$$\mathrm{i}\hat{e}\mathcal{K} \equiv X\mathcal{K} = 0 \ . \tag{3.49}$$

This is completely natural since in AdS the Kähler potential *must* be invariant – there are no Kähler transformations. Therefore the vector field X constructed is a Killing vector on the hyperkähler target space. Moreover, one can check that this vector field is actually tri-holomorphic. This follows from the fact that X is Hamiltonian with respect to the canonical holomorphic symplectic two-form $\omega = \mathrm{d}\Phi^I \wedge \mathrm{d}\Psi_I$.

In the super-Poincaré case, the complex variables $\varphi^a = (\Phi^I, \Psi_I)$ parametrize (an open domain of the zero section of) the cotangent bundle of $\mathcal{X}$. The curious feature of the AdS case is that Ψ_I does not transform as a (1,0) form at the point $(\Phi, \bar{\Phi})$ of $\mathcal{X}$. Indeed, let us consider two coordinate charts U and U' for $\mathcal{X}$ parametrized by complex coordinates Φ^I and $\Phi^{I'}$ respectively, which are related to each other by a holomorphic transformation $\Phi^{I'} = f^{I'}(\Phi)$, on the intersection of the charts, $U \bigcap U' \subset \mathcal{X}$. Let $K(\Phi, \bar{\Phi})$ and $K'(\Phi', \bar{\Phi}')$ be the Kähler potentials defined in the charts U and U' respectively. On the intersection $U \bigcap U' \subset \mathcal{X}$, we have

$$K'(\Phi', \bar{\Phi}') = K(\Phi, \bar{\Phi}) + \lambda(\Phi) + \bar{\lambda}(\bar{\Phi}) \ . \tag{3.50}$$

It can be seen that the variables Ψ_I and $\Psi_{I'}$ corresponding to the charts U and U' should be related to each other by the rule

$$\Psi_{I'} = \frac{\partial \Phi^J}{\partial \Phi^{I'}}\left(\Psi_J - \frac{\mathrm{i}\mu}{2|\mu|}\partial_J \lambda(\Phi)\right) \ , \tag{3.51}$$

in order to be consistent with the isometry transformation law (3.46). The transformation $(\Phi^I, \Psi_I) \to (\Phi^{I'}, \Psi_{I'})$ proves to be symplectic with respect to ω,

$$\mathrm{d}\Phi^I \wedge \mathrm{d}\Psi_I = \mathrm{d}\Phi^{I'} \wedge \mathrm{d}\Psi_{I'} \ . \tag{3.52}$$

One can think of (3.51) as the transformation law of a deformed holomorphic (1,0) form.

We can add some more flavor to the above discussion. First of all, let us consider the Kähler one-form on $\mathcal{X}$,

$$\rho = \frac{\mathrm{i}}{2}K_I \mathrm{d}\Phi^I - \frac{\mathrm{i}}{2}K_{\bar{I}}\mathrm{d}\bar{\Phi}^{\bar{I}} \ . \tag{3.53}$$

On the intersection of two charts, we have

$$\rho' = \rho + \frac{\mathrm{i}}{2}\mathrm{d}(\lambda - \bar{\lambda}) \ , \tag{3.54}$$

so ρ is not globally defined. Secondly, defining $\Psi := \Psi_I \mathrm{d}\Phi^I$ we observe that

$$\Psi' = \Psi - \frac{\mathrm{i}\mu}{2|\mu|}\mathrm{d}\lambda . \tag{3.55}$$

This result means that

$$\boldsymbol{\rho} := \rho + \frac{\bar{\mu}}{|\mu|}\Psi + \frac{\mu}{|\mu|}\bar{\Psi} \tag{3.56}$$

is a well-defined one-form on the hyperkähler target space.

3.2.2 A convenient fictitious coordinate

We would like to gauge the tri-holomorphic isometry introduced above using the intrinsic vector multiplet. This can be done using old results on gauged $\mathcal{N} = 1$ supersymmetric σ-models [42, 43, 44, 45, 46]. However, in order to gauge a Lagrangian, it is easiest to deal with *gauge invariant* Lagrangians. The standard way to deal with this, developed in [45], is to introduce a fictitious field which is a pure gauge degree of freedom that counters the gauge transformation of the original Lagrangian. That is, if K is the original polar multiplet Kähler potential transforming as

$$\mathrm{i}\hat{e}K = F(\Upsilon) + \check{F}(\check{\Upsilon}) , \tag{3.57}$$

we introduce a new arctic multiplet Υ^0 with the U(1) transformation law $\mathrm{i}\hat{e}\Upsilon^0 = F(\Upsilon)$ and consider the modified Kähler potential

$$K' = K - \Upsilon^0 - \check{\Upsilon}^0 . \tag{3.58}$$

This new Kähler potential is U(1) invariant. The theory with Lagrangian K' is invariant under gauge transformations

$$\Upsilon^0 \to \Upsilon^0 + \Lambda , \tag{3.59}$$

with Λ an arbitrary arctic superfield. This gauge symmetry allows us to gauge away Υ^0. In the gauge $\Upsilon^0 = 0$ we return to the original symmetry. In other words, the theory with Lagrangian K' is completely equivalent to the original σ-model.

Let us group all the arctic multiplets together as $\Upsilon^{I'} = (\Upsilon^0, \Upsilon^I)$. Their isometry transformation is generated by the holomorphic vector field $X^{I'} = (F, X^I)$ where both F and X^I depend only on Υ^I. On the tangent bundle coordinates, we find

$$\mathrm{i}\hat{e}\Phi^{I'} = X^{I'} \qquad \Longrightarrow \qquad \mathrm{i}\hat{e}\Phi^0 = F(\Phi) , \qquad \mathrm{i}\hat{e}\Phi^I = X^I(\Phi) , \tag{3.60a}$$

$$\mathrm{i}\hat{e}\Sigma^{I'} = \Sigma^{J'}\partial_{J'}X^{I'} \qquad \Longrightarrow \qquad \mathrm{i}\hat{e}\Sigma^0 = \Sigma^J\partial_J F(\Phi) , \qquad \mathrm{i}\hat{e}\Sigma^I = \Sigma^J\partial_J X^I(\Phi) , \tag{3.60b}$$

where $F(\Phi)$ and $X^I(\Phi)$ depend only on Φ^I.

Making use of K' leads to a modified Lagrangian $\mathcal{L}'$:

$$\mathcal{L}' = \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathcal{S}^{[2]}K' = \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathcal{S}^{[2]}(K - \Upsilon^0 - \check{\Upsilon}^0) = \mathcal{L} - \frac{\mathrm{i}\mu}{2|\mu|}\Sigma^0 + \frac{\mathrm{i}\bar{\mu}}{2|\mu|}\bar{\Sigma}^0 \ . \quad (3.61)$$

By construction $\mathcal{L}'$ is U(1) invariant, but we can see this explicitly by noting that the transformation of $\mathcal{L}$ is cancelled by the transformation of the additional terms.

Now we construct $\mathcal{K}' = \mathcal{L}' + \Sigma^{I'}\Psi_{I'} + \bar{\Sigma}^{\bar{J}'}\bar{\Psi}_{\bar{J}'}$. Because $\mathcal{L}'$ is invariant, we choose

$$\mathrm{i}\hat{e}\Psi_{I'} = -\partial_{I'}X^{J'}\Psi_{J'} \ . \quad (3.62)$$

This implies

$$\mathrm{i}\hat{e}\Psi_0 = 0 \ , \qquad \mathrm{i}\hat{e}\Psi_I = -\partial_I F\Psi_0 - \partial_I\Sigma^J\psi_J \ . \quad (3.63)$$

Now let us make an observation:

$$\mathcal{K}' = \mathcal{L} + \Sigma^I\Psi_I + \bar{\Sigma}^{\bar{I}}\bar{\Psi}_{\bar{I}} + \Sigma^0\left(\Psi_0 - \frac{\mathrm{i}\mu}{2|\mu|}\right) + \bar{\Sigma}^0\left(\bar{\Psi}_0 + \frac{\mathrm{i}\bar{\mu}}{2|\mu|}\right) \ . \quad (3.64)$$

Because $\mathcal{L}$ is independent of Σ^0, the equation of motion of Σ^0 merely enforces $\Psi_0 = \mathrm{i}\mu/2|\mu|$. This is consistent with $\mathrm{i}\hat{e}\Psi_0 = 0$. For the other Σ^I coordinates, we dualize as usual and end up with $\mathcal{K}' = \mathcal{K}$. The new Kähler potential is the same as the old! By construction it is gauge invariant. As before, we stay with the dynamical variables (Φ^I, Ψ_I) with the U(1) transformation law (3.48). There are no other dynamical fields.

3.3   Gauged isometries

Now let us gauge the Lagrangian $K'(\Upsilon, \check{\Upsilon})$ by covariantizing the arctic and antarctic superfields by the introduction of new covariant derivatives $\mathcal{D}_A$, eq. (2.1), where the vector multiplet associated with the generator $\hat{e}$ is the intrinsic vector multiplet of AdS, as discussed in section 2. Instead of ordinary arctic multiplets $\Upsilon^{I'}$ obeying the analyticity conditions (1.18), we have to consider gauge covariantly arctic multiplets constrained by

$$\mathcal{D}_\alpha^{(1)}\Upsilon^{I'} = \bar{\mathcal{D}}_{\dot\alpha}^{(1)}\Upsilon^{I'} = 0 \ , \qquad \mathcal{D}_\alpha^{(1)} := v_i\mathcal{D}_\alpha^i \ , \quad \bar{\mathcal{D}}_{\dot\alpha}^{(1)} := v_i\bar{\mathcal{D}}_{\dot\alpha}^i \ , \quad (3.65)$$

and similarly for their smile-conjugates $\check{\Upsilon}^{\bar{I}'}$. The gauge covariant superfields $\Upsilon^{I'}$ and $\check{\Upsilon}^{\bar{I}'}$ are assumed to have the functional form (1.24) and (1.25) respectively.

By the usual argument, one can show that $\Phi^{I'} = \Upsilon_0^{I'}$ is covariantly chiral. However, $\Sigma^{I'} = \Upsilon_1^{I'}$ is no longer complex linear, but is instead *modified* complex linear,

$$-\frac{1}{4}(\bar{\mathcal{D}}^2 - 4\mu)\Sigma^{I'} = X^{I'} . \tag{3.66}$$

The transformation law of the arctic multiplet

$$\delta\Upsilon^{I'} = -(\xi^A\mathcal{D}_A + \frac{1}{2}\lambda^{cd}M_{cd} + 2\varepsilon\mathcal{S}^{ij}J_{ij} + 2\mathrm{i}\varepsilon\hat{e})\Upsilon^{I'} \tag{3.67}$$

leads to the transformation laws of $\Phi^{I'}$ and $\Sigma^{I'}$:

$$\delta\Phi^{I'} = \bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\Sigma^{I'} + 2\varepsilon\mu\Sigma^{I'} - 2\mathrm{i}\varepsilon\hat{e}\Phi^{I'} \tag{3.68a}$$

$$= \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\Sigma^{I'}) \tag{3.68b}$$

and

$$\delta\Sigma^{I'} = -\varepsilon^\alpha\mathcal{D}_\alpha\Phi^{I'} + \bar{\varepsilon}_{\dot{\alpha}}\bar{\mathcal{D}}^{\dot{\alpha}}\Upsilon_2^{I'} + 4\mu\varepsilon\Upsilon_2^{I'} - 2\mathrm{i}\varepsilon\hat{e}\Sigma^{I'}$$

$$= -\varepsilon^\alpha\mathcal{D}_\alpha\Phi^{I'} + \bar{\mathcal{D}}_{\dot{\alpha}}(\bar{\varepsilon}^{\dot{\alpha}}\Upsilon_2^{I'}) - 2\mathrm{i}\varepsilon\hat{e}\Sigma^{I'} . \tag{3.69}$$

As before, we have $\mathrm{i}\hat{e}\Phi^{I'} = X^{I'}$ and $\mathrm{i}\hat{e}\Sigma^{I'} = \Sigma^{J'}\partial_{J'}X^{I'}$.

The tangent bundle Lagrangian is

$$\mathcal{L}' = \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathcal{S}^{[2]}K' . \tag{3.70}$$

We introduce the first order action

$$S_{\text{F.O.}} = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\mathcal{K}' - \left(\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\,\Psi_{I'}X^{I'} + \text{c.c.}\right) , \tag{3.71}$$

where

$$\mathcal{K}' = \mathcal{L}' + \Sigma^{I'}\Psi_{I'} + \bar{\Sigma}^{\bar{J}'}\bar{\Psi}_{\bar{J}'} . \tag{3.72}$$

The additional holomorphic terms we have added are necessary so that the equation of motion for $\Psi_{I'}$ imposes the modified complex linear condition on $\Sigma^{I'}$. We must take $\mathrm{i}\hat{e}\Psi_{I'} = -\partial_{I'}\Sigma^{J'}(\Phi)\Psi_{J'}$ so that each of the terms above is separately gauge invariant.

Next, we must determine the transformation rules. As before, $\delta\Phi^{I'}$ must be manifestly covariantly chiral,

$$\delta\Phi^{I'} := \frac{1}{2}(\bar{\mathcal{D}}^2 - 4\mu)(\varepsilon\Sigma^{I'}) . \tag{3.73}$$

This can be rewritten

$$\delta\Phi^{I'} := \bar{\varepsilon}_{\dot{\alpha}}\bar{\boldsymbol{\mathcal{D}}}^{\dot{\alpha}}\Sigma^{I'} + 2\varepsilon\mu\Sigma^{I'} + \frac{1}{2}\varepsilon(\bar{\boldsymbol{\mathcal{D}}}^2 - 4\mu)\Sigma^{I'} \ . \tag{3.74}$$

We keep the same rule for $\delta\Sigma^{I'}$. This leads to

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\delta\mathcal{L}'$$
$$= \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\left(\frac{1}{2}\Sigma^{I'}(\bar{\boldsymbol{\mathcal{D}}}^2 - 4\mu)\left(\varepsilon\frac{\partial\mathcal{L}'}{\partial\Phi^{I'}}\right) - 2\varepsilon\mathrm{i}\hat{e}\Sigma^{I'}\frac{\partial\mathcal{L}'}{\partial\Sigma^{I'}} + \mathrm{c.c.}\right) \ . \tag{3.75}$$

We postulate that

$$\delta\Psi_{I'} = -\frac{1}{2}(\bar{\boldsymbol{\mathcal{D}}}^2 - 4\mu)\left(\varepsilon\frac{\partial\mathcal{L}'}{\partial\Phi^{I'}}\right) \ , \tag{3.76}$$

which leads to

$$\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\delta\left(\mathcal{L}' + \Sigma^{I'}\Psi_{I'} + \bar{\Sigma}^{J'}\bar{\Psi}_{J'}\right)$$
$$= \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\left(-2\varepsilon\mathrm{i}\hat{e}\Sigma^{I'}\frac{\partial\mathcal{L}'}{\partial\Sigma^{I'}} - 2\varepsilon\mathrm{i}\hat{e}\Sigma^{I'}\Psi_{I'} + \mathrm{c.c.}\right) \ . \tag{3.77}$$

For the superpotential piece, we have

$$-\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathcal{E}\,\delta(\Psi_{I'}X^{I'}) = \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\left(-2\varepsilon\frac{\partial\mathcal{L}'}{\partial\Phi^{I'}}X^{I'} + 2\varepsilon\Psi_{I'}\partial_{J'}X^{I'}\Sigma^{J'}\right) \ . \tag{3.78}$$

Adding everything together gives

$$\delta S_{\mathrm{F.O.}} = -2\int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{d}^2\bar{\theta}\,E\,\varepsilon\,\mathrm{i}\hat{e}\mathcal{L}' = 0 \ . \tag{3.79}$$

Now we want to finish the duality by eliminating $\Sigma^{I'}$. Recall that

$$\mathcal{L}' = \frac{1}{2s}\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta}\mathcal{S}^{[2]}(K - \Upsilon^0 - \check{\Upsilon}^0) = \mathcal{L} - \frac{\mathrm{i}\mu}{2|\mu|}\Sigma^0 + \frac{\mathrm{i}\bar{\mu}}{2|\mu|}\bar{\Sigma}^0 \ . \tag{3.80}$$

Here $\mathcal{L}$ is independent of Σ^0 and Φ^0. Moreover, Φ^0 is completely absent from the action. The dual kinetic Lagrangian is

$$\mathcal{K}' = \mathcal{K} + \Sigma^0\left(\Psi_0 - \frac{\mathrm{i}\mu}{2|\mu|}\right) + \bar{\Sigma}^0\left(\bar{\Psi}_0 + \frac{\mathrm{i}\bar{\mu}}{2|\mu|}\right) \ . \tag{3.81}$$

The equation of motion for Σ^0 again fixes $\Psi_0 = \mathrm{i}\mu$, so we find $\mathcal{K}' = \mathcal{K}$. The elimination of the physical Σ^I proceeds as usual. However, now we have a superpotential,

$$W = -\Psi_{I'}X^{I'} = -\frac{\mathrm{i}\mu}{2|\mu|}F - \Psi_I X^I \ . \tag{3.82}$$

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So the full dual action can be written in the form

$$S_{\text{dual}} = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \left(\mathcal{K} + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}} \right) . \tag{3.83}$$

It is invariant under the extended supersymmetry transformation

$$\delta\Phi^I = +\frac{1}{2}(\bar{\boldsymbol{\mathcal{D}}}^2 - 4\mu)\left(\varepsilon \frac{\partial \mathcal{K}}{\partial \Psi_I} \right) , \qquad \delta\Psi_I = -\frac{1}{2}(\bar{\boldsymbol{\mathcal{D}}}^2 - 4\mu)\left(\varepsilon \frac{\partial \mathcal{K}}{\partial \Phi^I} \right) . \tag{3.84}$$

Some comments are in order. First, we have written the transformation laws in terms of the gauge-covariant $\mathcal{N} = 1$ derivatives $\boldsymbol{\mathcal{D}}_A$. However, because we chose the intrinsic vector multiplet to gauge the U(1) group, in this AdS frame the $\mathcal{N} = 1$ AdS derivatives possess no U(1) curvature. In other words, the U(1) connection is *pure gauge* and we can adopt a gauge where it vanishes, $\boldsymbol{\mathcal{D}}_A \to \mathcal{D}_A$. This removes all trace of the gauging from the $\mathcal{N} = 1$ superspace geometry.

Second, the tri-holomorphic isometry $X^a = (X^I, X_I)$ with

$$X^I = \mathrm{i}\hat{e}\Phi^I = X^I(\Phi) = -\frac{\partial W}{\partial \Psi_I} ,$$

$$X_I = \mathrm{i}\hat{e}\Psi_I = -\frac{\mathrm{i}\mu}{2|\mu|}F_I - \partial_I X^J \Psi_J = +\frac{\partial W}{\partial \Phi_I} , \tag{3.85}$$

indeed obeys $X^a = -\omega^{ab}W_b$, as required.

4 Poincaré coordinates for AdS$^{4|4}$ and AdS$^{4|8}$

The aim of this section is to describe a new conformally flat realization for four-dimensional (4D) $\mathcal{N} = 2$ AdS superspace with the property that AdS$^{4|8}$ is foliated into a union of 3D $\mathcal{N} = 4$ flat superspaces with a real central charge corresponding to a derivative in the fourth dimension. This realization will be used in the next sections. As a warm-up exercise, we first consider the case of 4D $\mathcal{N} = 1$ AdS superspace.

4.1 AdS$^{4|4}$

The conformal flatness of the superspace AdS$^{4|4}$ was established by Ivanov and Sorin [47] and later on reviewed, in the modern form, in textbooks [48, 49]. The approaches

pursued in [47, 48, 49] made use of stereographic coordinates in AdS_4 in which the space-time metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^a \mathrm{d}\,x_a}{\left(1 - \frac{1}{4}|\mu|^2 x^2\right)^2} \; . \tag{4.1}$$

This metric is manifestly invariant under the group of four-dimensional Lorentz transformations, O(3,1). Here we would like to derive an alternative conformally flat realization of $\text{AdS}^{4|4}$ which is characterized by the space-time metric (1.4). The latter metric is invariant under the group of three-dimensional Poincaré transformations, IO(2,1).

Let us start by recalling the structure of the super-Weyl transformations in 4D $\mathcal{N} = 1$ old minimal supergravity [50]. The superspace geometry of supergravity is described by covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot\alpha}) = E_A{}^M \partial_M + \frac{1}{2}\Omega_A{}^{bc} M_{bc} \tag{4.2}$$

and a set of constrained superfields, $\mathcal{R}$, $\mathcal{G}_{\alpha\dot\alpha}$ and $\mathcal{W}_{\alpha\beta\gamma}$ in terms of which the torsion and curvature tensors are constructed [50]. We refer the reader to [49] for a detailed description of the geometry of old minimal supergravity. Let $D_A = (D_a, D_\alpha, \bar{D}^{\dot\alpha})$ be another set of superspace covariant derivatives characterized by the torsion superfields R, $G_{\alpha\dot\alpha}$ and $W_{\alpha\beta\gamma}$. The two supergeometries, which are associated with $\mathcal{D}_A$ and D_A, are said to be conformally related if their covariant derivatives are related by a super-Weyl transformation of the form[13] [51]

$$\mathcal{D}_\alpha = \mathrm{e}^{\frac{1}{2}\sigma - \bar\sigma}\left(D_\alpha - (D^\beta\sigma)M_{\alpha\beta}\right) , \tag{4.3a}$$

$$\bar{\mathcal{D}}_{\dot\alpha} = \mathrm{e}^{\frac{1}{2}\bar\sigma - \sigma}\left(\bar{D}_{\dot\alpha} - (\bar{D}^{\dot\beta}\bar\sigma)\bar{M}_{\dot\alpha\dot\beta}\right) , \tag{4.3b}$$

$$\mathcal{D}_{\alpha\dot\beta} = \frac{\mathrm{i}}{2}\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\beta}\} , \tag{4.3c}$$

where the parameter σ is covariantly chiral, $\bar{D}_{\dot\alpha}\sigma = 0$. The components of the torsion transform as

$$\mathcal{R} = -\frac{1}{4}\mathrm{e}^{-2\sigma}(\bar{D}^2 - 4R)\mathrm{e}^{\bar\sigma} , \tag{4.4a}$$

$$\mathcal{G}_{\alpha\dot\alpha} = \mathrm{e}^{-(\sigma-\bar\sigma)/2}\left(G_{\alpha\dot\alpha} + \frac{1}{2}(D_\alpha\sigma)(\bar{D}_{\dot\alpha}\bar\sigma) + \mathrm{i}D_{\alpha\dot\alpha}(\bar\sigma - \sigma)\right) , \tag{4.4b}$$

$$\mathcal{W}_{\alpha\beta\gamma} = \mathrm{e}^{-3\sigma/2}W_{\alpha\beta\gamma} . \tag{4.4c}$$

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[13]Here $M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta}M_{ab}$, $\bar{M}_{\dot\alpha\dot\beta} = -\frac{1}{2}(\tilde\sigma^{ab})_{\dot\alpha\dot\beta}M_{ab}$ and M_{ab} are the Lorentz generators with spinor and vector indices, respectively, see [49] for more details.

If the covariant derivatives D_A are flat, and hence $R = G_{\alpha\dot\alpha} = W_{\alpha\beta\gamma} = 0$, then the geometry described by $\mathcal{D}_A$ is said to be conformally flat. A well-known example of conformally flat supergeometry is $\mathrm{AdS}^{4|4}$. The geometry of this superspace is characterized by $\mathcal{G}_{\alpha\dot\alpha} = \mathcal{W}_{\alpha\beta\gamma} = 0$ and $\mathcal{R} = \mu = \mathrm{const}$, $\mu \neq 0$. The covariant derivatives obey the (anti-) commutation relations (1.8). The requirement that $\mathrm{AdS}^{4|4}$ is conformally flat means that there exists a 'flat' chiral superfield σ, $\bar{D}_{\dot\alpha}\sigma = 0$, such that

$$\mu = -\frac{1}{4}\mathrm{e}^{-2\sigma}\bar{D}^2\mathrm{e}^{\bar\sigma} \ , \tag{4.5a}$$

$$0 = \frac{1}{2}(D_\alpha\sigma)(\bar{D}_{\dot\alpha}\bar\sigma) + \mathrm{i}\partial_{\alpha\dot\alpha}(\bar\sigma - \sigma) \ , \tag{4.5b}$$

where $D_A = (\partial_a, D_\alpha, \bar{D}^{\dot\alpha})$ are the covariant derivatives of 4D $\mathcal{N} = 1$ Minkowski superspace parametrized by Cartesian coordinates $(x^a, \theta^\alpha, \bar\theta_{\dot\alpha})$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}\bar\theta^{\dot\beta}\partial_{\alpha\dot\beta} \ , \qquad \bar{D}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \mathrm{i}\theta^\beta\partial_{\beta\dot\alpha} \ . \tag{4.6}$$

Note that the equation (4.5b) can be equivalently rewritten as

$$0 = [D_\alpha, \bar{D}_{\dot\beta}]\mathrm{e}^{-\frac{1}{2}(\sigma+\bar\sigma)} \ . \tag{4.7}$$

At this stage, it is convenient to introduce a $3 + 1$ splitting of the 4D vector indices that is suitable for a 3D foliation of AdS_4. We adopt the 3D spinor notation introduced in [52, 53]. The 4D sigma-matrices are

$$(\sigma_a)_{\alpha\dot\beta} := (\mathbb{1}, \vec\sigma) \ , \qquad (\tilde\sigma_a)^{\dot\alpha\beta} := \varepsilon^{\beta\gamma}\varepsilon^{\dot\alpha\dot\delta}(\sigma_a)_{\gamma\dot\delta} = (\mathbb{1}, -\vec\sigma) \ , \qquad m = 0, 1, 2, 3 \ , \tag{4.8}$$

where $\vec\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. By deleting the matrices with space index $a = 2$, we obtain the real and symmetric 3D gamma-matrices

$$(\sigma_a)_{\alpha\dot\beta} \quad \longrightarrow \quad (\gamma_{\hat a})_{\alpha\beta} = (\gamma_{\hat a})_{\beta\alpha} = (\mathbb{1}, \sigma_1, \sigma_3) \ , \tag{4.9a}$$

$$(\tilde\sigma_a)^{\dot\alpha\beta} \quad \longrightarrow \quad (\gamma_{\hat a})^{\alpha\beta} = (\gamma_{\hat a})^{\beta\alpha} = \varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}(\gamma_{\hat a})_{\gamma\delta} \ , \tag{4.9b}$$

where the spinor indices are raised and lowered using the $\mathrm{SL}(2,\mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ , \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta^\alpha_\beta \tag{4.10}$$

by the rule:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \ , \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \ . \tag{4.11}$$

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Note that the 3D gamma-matrices satisfy[14]

$$\gamma_{\hat{a}} := (\gamma_{\hat{a}})_{\alpha}{}^{\beta} = \varepsilon^{\beta\gamma}(\gamma_{\hat{a}})_{\alpha\gamma} , \tag{4.12a}$$

$$\{\gamma_{\hat{a}}, \gamma_{\hat{b}}\} = 2\eta_{\hat{a}\hat{b}}\mathbb{1} , \qquad \gamma_{\hat{a}}\gamma_{\hat{b}} = \eta_{\hat{a}\hat{b}}\mathbb{1} + \varepsilon_{\hat{a}\hat{b}\hat{c}}\gamma^{\hat{c}} . \tag{4.12b}$$

There is no difference between dotted and undotted spinor indices in three dimensions. Given a four-vector V_a, it decomposes as follows

$$V_{\alpha\dot{\beta}} = (\sigma^a)_{\alpha\dot{\beta}}V_a \rightarrow V_{\alpha\beta} + \mathrm{i}\,\varepsilon_{\alpha\beta}V_z , \qquad V_{\alpha\beta} := (\gamma^{\hat{a}})_{\alpha\beta}V_{\hat{a}} , \tag{4.13a}$$

$$V^{\dot{\alpha}\beta} = (\tilde{\sigma}^a)^{\dot{\alpha}\beta}V_a \rightarrow V^{\alpha\beta} + \mathrm{i}\,\varepsilon^{\alpha\beta}V_z , \qquad V^{\alpha\beta} := (\gamma^{\hat{a}})^{\alpha\beta}V_{\hat{a}} , \tag{4.13b}$$

where $\hat{a} = 0, 1, 3$. In particular, the 4D vector coordinates x^a split as $x^a = (x^{\hat{a}}, z)$, where $z := x^2$. Then, the 4D $\mathcal{N} = 1$ flat covariant derivatives take the form

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + \mathrm{i}\bar{\theta}^{\beta}\partial_{\alpha\beta} - \bar{\theta}_{\alpha}\partial_z , \quad \bar{D}_{\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\alpha}} - \mathrm{i}\theta^{\beta}\partial_{\alpha\beta} - \theta_{\alpha}\partial_z . \tag{4.14}$$

They obey the anti-commutation relations

$$\{D_{\alpha}, D_{\beta}\} = \{\bar{D}_{\alpha}, \bar{D}_{\beta}\} = 0 , \qquad \{D_{\alpha}, \bar{D}_{\beta}\} = -2\mathrm{i}\partial_{\alpha\beta} + 2\varepsilon_{\alpha\beta}\partial_z \tag{4.15}$$

which correspond to the 3D $\mathcal{N} = 2$ Poincaré supersymmetry with central charge. The central charge is identified with ∂_z.

We are now prepared to look for a solution of the equations (4.5a) and (4.7). Let us define the bosonic coordinates

$$z_L = z - \theta^{\alpha}\bar{\theta}_{\alpha} , \qquad z_R = z + \theta^{\alpha}\bar{\theta}_{\alpha} , \tag{4.16}$$

which are respectively chiral, $\bar{D}_{\alpha}z_L = 0$, and antichiral, $D_{\alpha}z_R = 0$, with respect to the derivatives (4.14). Then, it is a short computation to prove that the superfields

$$\mathrm{e}^{-\sigma} = |\mu|z_L - \bar{\mu}\theta^2 , \qquad \mathrm{e}^{-\bar{\sigma}} = |\mu|z_R - \mu\bar{\theta}^2 \tag{4.17}$$

satisfy the equations (4.5a) and (4.7). As a result, the relations (4.3a)–(4.3c) with $\mathrm{e}^{-\sigma}$ given above define a conformally flat realization of $\mathrm{AdS}^{4|4}$.

Given a superfield $U(x^{\hat{a}}, z, \theta^{\alpha}, \bar{\theta}_{\alpha})$, we introduce the projection $U| := U(x^{\hat{a}}, z, 0, 0)$. For the covariant derivatives the projection is defined similarly

$$\mathcal{D}_A| := E_A{}^M|\partial_M + \frac{1}{2}\Phi_A{}^{bc}|M_{bc} . \tag{4.18}$$

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[14] The 3D Minkowski metric is $\eta_{mn} = \eta^{mn} = \mathrm{diag}(-1, 1, 1)$ and the Levi-Civita tensor is normalised as $\varepsilon_{012} = -\varepsilon^{012} = -1$.

It follows that

$$\nabla_a := \mathcal{D}_a| = \mathrm{e}^{-\frac{1}{2}(\sigma+\bar{\sigma})}|\partial_a + \cdots = |\mu|z\,\partial_a + \cdots \qquad (4.19)$$

where the ellipses denote the Lorentz connection. For the vierbein we get

$$e_a{}^m = |\mu|z\,\delta_a^m \;. \qquad (4.20)$$

Therefore the space-time metric has the form (1.4) with $s = |\mu|$.

4.2   AdS$^{4|8}$

The $\mathcal{N} = 2$ AdS superspace was briefly introduced in subsection 1.1. We recall that AdS$^{4|8}$ is a maximally symmetric geometry that originates within the superspace formulation of $\mathcal{N} = 2$ conformal supergravity developed in [11]. This formulation is based on the curved superspace geometry given by Grimm [54]. What makes this geometry suitable to describe conformal supergravity is the invariance of the corresponding constraints under certain super-Weyl transformations discovered in [11].

Let us summarize the main ingredients of the formulation of 4D $\mathcal{N} = 2$ conformal supergravity given in [11]. The superspace geometry is described by covariant derivatives of the form

$$\mathcal{D}_\mathcal{A} = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_i^{\dot{\alpha}}) = E_\mathcal{A}{}^\mathcal{M}\partial_\mathcal{M} + \frac{1}{2}\Omega_\mathcal{A}{}^{bc}M_{bc} + \Phi_\mathcal{A}{}^{kl}J_{kl} \;, \qquad (4.21)$$

where $E_\mathcal{A}{}^\mathcal{M}$ is the supervielbein, $\Omega_\mathcal{A}{}^{bc}$ the Lorentz connection and $\Phi_\mathcal{A}{}^{kl}$ the SU(2) connection (with J_{kl} being the corresponding generators). The covariant derivatives are subject to certain conventional constraints [54] which are solved in terms of several dimension-1 constrained superfields, $\mathcal{S}_{ij} = \mathcal{S}_{ji}$, $\mathcal{G}_{\alpha\dot{\alpha}}$, $\mathcal{Y}_{\alpha\beta} = \mathcal{Y}_{\beta\alpha}$ and $\mathcal{W}_{\alpha\beta} = \mathcal{W}_{\beta\alpha}$, and their covariant derivatives. The superfield $\mathcal{G}_{\alpha\dot{\alpha}}$ is real, $\overline{\mathcal{G}_{\alpha\dot{\alpha}}} = \mathcal{G}_{\alpha\dot{\alpha}}$, while the other torsion components are in general complex (the torsion $\mathcal{S}_{ij}$ can be made real in a special super-Weyl gauge). The constraints turn out to be invariant under super-Weyl transformations generated by a covariantly chiral parameter σ. The super-Weyl transformations were given originally in [11] in the infinitesimal form, and then in [10] in the finite form.

Two superspace geometries described by covariant derivatives $\mathcal{D}_\mathcal{A} = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_i^{\dot{\alpha}})$ and $D_\mathcal{A} = (D_a, D_\alpha^i, \bar{D}_i^{\dot{\alpha}})$ are conformally related if the covariant derivatives $\mathcal{D}_\mathcal{A}$ are obtained

from $D_\mathcal{A}$ by applying a finite super-Weyl transformation

$$\mathcal{D}_\alpha^i = e^{\frac{1}{2}\bar\sigma}\left(D_\alpha^i + (D^{\gamma i}\sigma)M_{\gamma\alpha} - (D_{\alpha k}\sigma)J^{ki}\right) , \tag{4.22a}$$

$$\bar{\mathcal{D}}_{\dot\alpha i} = e^{\frac{1}{2}\sigma}\left(\bar D_{\dot\alpha i} + (\bar D_i^{\dot\gamma}\bar\sigma)\bar M_{\dot\gamma\dot\alpha} + (\bar D_{\dot\alpha}^k\bar\sigma)J_{ki}\right) , \tag{4.22b}$$

$$\mathcal{D}_a = e^{\frac{1}{2}(\sigma+\bar\sigma)}\left(D_a + \frac{i}{4}(\sigma_a)^\alpha{}_{\dot\beta}(\bar D_k^{\dot\beta}\bar\sigma)D_\alpha^k + \frac{i}{4}(\sigma_a)^\alpha{}_{\dot\beta}(D_\alpha^k\sigma)\bar D_k^{\dot\beta} - \frac{1}{2}\left(D^b(\sigma+\bar\sigma)\right)M_{ab}\right.$$
$$+ \frac{i}{8}(\tilde\sigma_a)^{\dot\alpha\alpha}(D^{\beta k}\sigma)(\bar D_{\dot\alpha k}\bar\sigma)M_{\alpha\beta} + \frac{i}{8}(\tilde\sigma_a)^{\dot\alpha\alpha}(\bar D_k^{\dot\beta}\bar\sigma)(D_\alpha^k\sigma)\bar M_{\dot\alpha\dot\beta}$$
$$\left. - \frac{i}{4}(\tilde\sigma_a)^{\dot\alpha\alpha}(D_\alpha^k\sigma)(\bar D_{\dot\alpha}^l\bar\sigma)J_{kl}\right) , \tag{4.22c}$$

where the parameter σ is covariantly chiral $\bar D_i^{\dot\alpha}\sigma = 0$. The dimension-1 components of the torsion in the two geometries are related to each other as follows:

$$\mathcal{S}_{ij} = e^{\bar\sigma}\left(S_{ij} - \frac{1}{4}(D_{(i}^\gamma D_{\gamma j)}\sigma) + \frac{1}{4}(D_{(i}^\gamma\sigma)(D_{\gamma j)}\sigma)\right) , \tag{4.23a}$$

$$\mathcal{G}_\alpha{}^{\dot\beta} = e^{\frac{1}{2}(\sigma+\bar\sigma)}\left(G_\alpha{}^{\dot\beta} - \frac{i}{4}(\sigma^c)_\alpha{}^{\dot\beta}D_c(\sigma-\bar\sigma) - \frac{1}{8}(D_\alpha^k\sigma)(\bar D_k^{\dot\beta}\bar\sigma)\right) , \tag{4.23b}$$

$$\mathcal{Y}_{\alpha\beta} = e^{\bar\sigma}\left(Y_{\alpha\beta} - \frac{1}{4}(D_{(\alpha}^k D_{\beta)k}\sigma) - \frac{1}{4}(D_{(\alpha}^k\sigma)(D_{\beta)k}\sigma)\right) , \tag{4.23c}$$

$$\mathcal{W}_{\alpha\beta} = e^\sigma W_{\alpha\beta} . \tag{4.23d}$$

The geometry described by $\mathcal{D}_\mathcal{A}$ is said to be *conformally flat* if the covariant derivatives $D_\mathcal{A}$ correspond to a flat superspace characterized by $\mathcal{S}_{ij} = \mathcal{G}_a = \mathcal{Y}_{\alpha\beta} = \mathcal{W}_{\alpha\beta} = 0$. An example of a conformally flat superspace is AdS$^{4|8}$ [10].[15] Its geometry is completely determined by a nonzero, real, covariantly constant isotriplet

$$\mathcal{S}^{ij} = \mathcal{S}^{ji} , \qquad \overline{\mathcal{S}^{ij}} = \mathcal{S}_{ij} , \qquad \mathcal{D}_\mathcal{A}\mathcal{S}^{jk} = 0 , \tag{4.24}$$

while the other component of the torsion vanish,

$$\mathcal{W}_{\alpha\beta} = \mathcal{Y}_{\alpha\beta} = \mathcal{G}_{\alpha\dot\beta} = 0 . \tag{4.25}$$

The covariant derivatives of AdS$^{4|8}$ obey the (anti-)commutation relations (1.2). The proof of the conformal flatness of AdS$^{4|8}$ given in [10] was based on the use of the stereographic coordinates for AdS$_4$. We now derive a new conformally flat realization for AdS$^{4|8}$ that makes use of the Poincaré coordinates.

The condition of conformal flatness of AdS$^{4|8}$ means that there exists a chiral superfield σ such that $\mathcal{Y}_{\alpha\beta} = \mathcal{G}_{\alpha\dot\beta} = 0$ and $\mathcal{S}^{ij} = \bar{\mathcal{S}}^{ij}$ is covariantly constant. Let $D_\mathcal{A} = (\partial_a, D_\alpha^i, \bar D_i^{\dot\alpha})$ be the covariant derivatives of the $\mathcal{N} = 2$ Minkowski superspace,

$$D_\alpha^i = \frac{\partial}{\partial\theta_i^\alpha} - i(\sigma^b)_\alpha{}^{\dot\beta}\bar\theta_{\dot\beta}^i\partial_b , \qquad \bar D_i^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}^i} - i(\sigma^b)_\beta{}^{\dot\alpha}\theta_i^\beta\partial_b , \tag{4.26}$$

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[15] In the framework of nonlinear realizations, the conformal flatness of AdS$^{4|8}$ was shown in [55].

with the standard anti-commutation relations

$$\{D_\alpha^i, D_\beta^j\} = \{\bar{D}_i^{\dot{\alpha}}, \bar{D}_j^{\dot{\beta}}\} = 0 \ , \qquad \{D_\alpha^i, \bar{D}_j^{\dot{\beta}}\} = -2\delta_j^i (\sigma^a)_\alpha{}^{\dot{\beta}} \partial_a \ . \tag{4.27}$$

It follows from (4.23b) that the equation $\mathcal{G}_{\alpha\dot{\beta}} = 0$ is equivalent to

$$[D_\alpha^k, \bar{D}_k^{\dot{\alpha}}] \mathrm{e}^{\sigma + \bar{\sigma}} = 0 \ . \tag{4.28}$$

In accordance with (4.23c), the condition $\mathcal{Y}_{\alpha\beta} = 0$ is equivalent to

$$D_{(\alpha}^k D_{\beta)k} \, \mathrm{e}^\sigma = 0 \ . \tag{4.29}$$

The equation (4.23a) leads to

$$\mathcal{S}^{ij} = \frac{1}{4} \mathrm{e}^{\sigma + \bar{\sigma}} (D^{ij} \mathrm{e}^{-\sigma}) = \frac{1}{4} \mathrm{e}^{\sigma + \bar{\sigma}} (\bar{D}^{ij} \mathrm{e}^{-\sigma}) \ , \tag{4.30}$$

where $D^{ij} := D^{\alpha i} D_\alpha^j = D^{ji}$ and $\bar{D}^{ij} := \bar{D}_{\dot{\alpha}}^i \bar{D}^{\dot{\alpha} j} = \bar{D}^{ji}$. Due to (4.23d), the equation $\mathcal{W}_{\alpha\beta} = 0$ is satisfied automatically. The condition (4.30) tells us that the chiral superfield

$$W := \mathrm{e}^{-\sigma} \ , \qquad \bar{D}_i^{\dot{\alpha}} W = 0 \tag{4.31}$$

obeys the reality condition

$$D^{ij} W = \bar{D}^{ij} \bar{W} \ , \tag{4.32}$$

which is the Bianchi identity for the chiral field strength, W, of an Abelian vector multiplet in flat superspace [33]. Associated with this vector multiplet in flat superspace is the *intrinsic vector multiplet* in AdS$^{4|8}$ (introduced in section 2) such that its covariantly chiral field strength, $\mathcal{W}$, is constant,

$$\mathcal{W} = \mathrm{e}^\sigma W = 1 \ , \tag{4.33}$$

where we have used the super-Weyl transformation law of the vector multiplet [11]. It should also be mentioned that the Bianchi identity (4.32) implies that the real isotriplet

$$\Sigma^{ij} := \frac{1}{4} D^{ij} W \ , \qquad \overline{\Sigma^{ij}} := \frac{1}{4} \bar{D}_{ij} \bar{W} = \varepsilon_{ik} \varepsilon_{jl} \Sigma^{kl} \tag{4.34}$$

satisfies the constraints

$$D_\alpha^{(i} \Sigma^{jk)} = \bar{D}_{\dot{\alpha}}^{(i} \Sigma^{jk)} = 0 \tag{4.35}$$

which are characteristic of the $\mathcal{N} = 2$ linear multiplet. The above relations are completely general in the sense that they hold for any super-Weyl parameter σ which conformally relates AdS$^{4|8}$ to flat superspace.

Now, we turn to deriving an IO(2,1)-invariant solution of the equations (4.29) and (4.30) which leads to the Poincaré coordinates for AdS_4. In complete analogy with the $\mathcal{N} = 1$ case described in the previous subsection, we introduce a 3D foliation of the space-time coordinates. Then, the 4D $\mathcal{N} = 2$ flat covariant derivatives take the form

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + i(\gamma^{\hat{m}})_{\alpha\beta}\bar{\theta}^{\beta i}\partial_{\hat{m}} - \bar{\theta}^i_\alpha\partial_z \ , \quad \bar{D}_{\alpha i} = -\frac{\partial}{\partial\bar{\theta}^{\alpha i}} - i(\gamma^{\hat{m}})_{\alpha\beta}\theta^\beta_i\partial_{\hat{m}} - \theta_{\alpha i}\partial_z \ . \quad (4.36)$$

The anti-commutation relations for the covariant derivatives turn into

$$\{D^i_\alpha, D^j_\beta\} = \{\bar{D}_{\alpha i}, \bar{D}_{\beta j}\} = 0 \ , \quad \{D^i_\alpha, \bar{D}_{\beta j}\} = -2i\delta^i_j(\gamma^{\hat{m}})_{\alpha\beta}\partial_{\hat{m}} + 2\delta^i_j\varepsilon_{\alpha\beta}\partial_z \ . \quad (4.37)$$

These relations correspond to the 3D $\mathcal{N} = 4$ super-Poincaré algebra with a real central charge.

We make the most general IO(2,1)-invariant ansatz for σ in (4.22)

$$e^\sigma := A(z_L) + \theta_{ij}B^{ij}(z_L) + \theta^{ij}\theta_{ij}C(z_L) \ , \qquad \theta_{ij} := \theta^\alpha_i\theta_{\alpha j} \ , \qquad (4.38)$$

where $z_L := z - \theta^\alpha_k\bar{\theta}^k_\alpha$ is the chiral completion of the space coordinate z, $\bar{D}_{\alpha i}z_L = 0$. The unknown functions $A(z_L)$, $B^{ij}(z_L)$ and $C(z_L)$ can be determined by requiring the equations (4.28), (4.29) and (4.30) to hold. The solution to these equations (compare with the five-dimensional case [16]) is

$$e^\sigma = s\,z_L + s^{ij}\theta_{ij} \ , \qquad (4.39)$$

where s^{ij} is a constant real isotriplet,

$$s^{ij} = \bar{s}^{ij} \ , \qquad\qquad s^2 := \frac{1}{2}s^{ij}s_{ij} \ . \qquad (4.40)$$

Evaluating the torsion superfield $\mathcal{S}^{ij}$ gives

$$\mathcal{S}^{ij} = \frac{1}{4}e^{\sigma+\bar{\sigma}}(D^{ij}e^{-\sigma}) = \frac{1}{4}e^{\sigma+\bar{\sigma}}(\bar{D}^{ij}e^{-\bar{\sigma}}) = s^{ij} + O(\theta) \ . \qquad (4.41)$$

It can be checked that $\mathcal{S}^{ij}$ is real, $\mathcal{S}^{ij} = \bar{\mathcal{S}}^{ij}$. It is covariantly constant by construction. It follows from (4.41) that

$$\mathcal{S}^2 := \frac{1}{2}\mathcal{S}^{ij}\mathcal{S}_{ij} = \frac{1}{2}s^{ij}s_{ij} = s^2 \ . \qquad (4.42)$$

This completes the derivation of the conformally flat representation for $AdS^{4|8}$. In what follows we do not distinguish between $\mathcal{S}$ and s.

It should be remarked that $\mathcal{S}^{ij}$ in (4.41) is covariantly constant, $\mathcal{D}_{\mathcal{A}}\mathcal{S}^{kl} = 0$, but not constant. The point is that the conformally flat representation for the covariant derivatives, eqs. (4.22a) and (4.22b), is given in terms of a linear combination of all the generators of the group $\mathrm{SU}(2)_R$. As discussed in subsection 1.1, the $\mathrm{SU}(2)$ gauge freedom can be used to bring the $\mathrm{SU}(2)$ connection of $\mathrm{AdS}^{4|8}$ to the form $\Phi_{\mathcal{A}}{}^{ij} = \Phi_{\mathcal{A}}\mathcal{S}^{ij}$, for some one-form $\Phi_{\mathcal{A}}$. In such a gauge, $\mathcal{S}^{ij}$ becomes constant, $\mathcal{S}^{ij} = s^{ij} = \mathrm{const}$.

For our analysis in the next sections, it is important to know explicit expressions for the components of Σ^{ij}, eq. (4.34). To derive them, it proves useful to replace Σ^{ij} with an index-free object $\Sigma^{(2)}(v)$ obtained by contracting the $\mathrm{SU}(2)$ indices of Σ^{ij} with an auxiliary bosonic isotwistor $v^i \in \mathbb{C}^2 \setminus \{0\}$:

$$\Sigma^{(2)}(v) := v_i v_j \Sigma^{ij} . \tag{4.43}$$

The significance of this definition will become clear later in the paper when the projective superspace techniques will play a central role. For now, let us focus on presenting some technical results concerning $\Sigma^{(2)}$.

In terms of $\Sigma^{(2)}$, the equations (4.35) take the form of analyticity constraints

$$D_\alpha^{(1)}\Sigma^{(2)} = \bar{D}_\alpha^{(1)}\Sigma^{(2)} = 0 , \tag{4.44}$$

where we have introduced

$$D_\alpha^{(1)} := v_i D_\alpha^i , \quad \bar{D}_\alpha^{(1)} := v_i \bar{D}_\alpha^i . \tag{4.45}$$

It follows from the anti-commutation relations (4.27) that the fermionic operators $D_\alpha^{(1)}$ and $\bar{D}_\alpha^{(1)}$ strictly anti-commute with each other,

$$\{D_\alpha^{(1)}, D_\beta^{(1)}\} = \{D_\alpha^{(1)}, \bar{D}_\beta^{(1)}\} = \{\bar{D}_\alpha^{(1)}, \bar{D}_\beta^{(1)}\} = 0 . \tag{4.46}$$

These properties allow us to define flat projective supermultiplets (compare with the definition given in subsection 1.3).

We introduce a new basis for the superspace variables z, θ_i^α and $\bar{\theta}_i^\alpha$ (which is analogous to the analytic basis in harmonic superspace [5]) defined as follows:

$$\theta_\alpha^{(1)} := v_i \theta_\alpha^i , \quad \theta_\alpha^{(-1)} := \frac{1}{(v,u)} u_i \theta_\alpha^i , \quad \bar{\theta}_\alpha^{(1)} := v_i \bar{\theta}_\alpha^i , \quad \bar{\theta}_\alpha^{(-1)} := \frac{1}{(v,u)} u_i \bar{\theta}_\alpha^i , \tag{4.47a}$$

$$z_A := z + \theta^{\alpha(1)}\bar{\theta}_\alpha^{(-1)} + \theta^{\alpha(-1)}\bar{\theta}_\alpha^{(1)} . \tag{4.47b}$$

Here we have introduced a second isotwistor $u_i \in \mathbb{C}^2 \backslash \{0\}$ which is subject to the condition that $(v, u) := v^i u_i = \varepsilon_{ij} v^i u^j \neq 0$, but otherwise is completely arbitrary. The coordinates $Z_A := (z_A, \theta_\alpha^{(1)}, \bar{\theta}_\alpha^{(1)})$ are annihilated by the derivatives $D_\alpha^{(1)}$ and $\bar{D}_\alpha^{(1)}$,

$$D_\alpha^{(1)} Z_A = \bar{D}_\alpha^{(1)} Z_A = 0 . \tag{4.48}$$

In terms of the variables Z_A, the superfield $\Sigma^{(2)}$ can be shown to have the form:

$$\begin{aligned}
\Sigma^{(2)} &= s^{-2}(z_A)^{-2} s^{(2)} - 2s^{-1}(z_A)^{-3} \left(\theta^{(2)} + \bar{\theta}^{(2)} \right) + 4s^{-2}(z_A)^{-3} s^{(0)} \theta^{\alpha(1)} \bar{\theta}_\alpha^{(1)} \\
&\quad + 6s^{-2}(z_A)^{-4} s^{(-2)} \theta^{(2)} \bar{\theta}^{(2)} ,
\end{aligned} \tag{4.49}$$

where $\theta^{(2)} = \theta^{\alpha(1)} \theta_\alpha^{(1)}$, $\bar{\theta}^{(2)} = \bar{\theta}_\alpha^{(1)} \bar{\theta}^{\alpha(1)}$ and

$$s^{(2)} := v_i v_j s^{ij} , \qquad s^{(0)} := \frac{v_i u_j}{(v, u)} s^{ij} , \qquad s^{(-2)} := \frac{u_i u_j}{(v, u)^2} s^{ij} . \tag{4.50}$$

The expression (4.49) makes manifest the fact that $\Sigma^{(2)}$ satisfies the constraints $D_\alpha^{(1)} \Sigma^{(2)} = \bar{D}_{\dot{\alpha}}^{(1)} \Sigma^{(2)} = 0$. Moreover, despite the fact that the separate contributions in the right-hand side of (4.49) explicitly depend on u_i, it is easy to prove that $\Sigma^{(2)}$ is independent of u_i,

$$\frac{\partial}{\partial u_i} \Sigma^{(2)} = 0 , \tag{4.51}$$

as it should be in accordance with (4.43).

5 Off-shell supersymmetric theories in AdS$^{4|8}$ using the 3D foliation

As discussed in subsection 1.3, general $\mathcal{N} = 2$ supersymmetric theories in AdS$_4$ can be formulated in terms of covariant projective supermultiplets living in the projective superspace AdS$^{4|8} \times \mathbb{C}P^1$. Here we would like to reformulate, following the five-dimensional analysis of [16], the dynamics of these theories in terms of flat projective supermultiplets by using the conformally flat realization of AdS$^{4|8}$ introduced in the previous section.

5.1 Supersymmetric action

The manifestly supersymmetric action principle in AdS$^{4|8} \times \mathbb{C}P^1$ is given by (1.23). Our goal is to bring this action to a flat superspace form which is based on the use of

the conformally flat realization of AdS$^{4|8}$ introduced in the previous section. In order to achieve this, the key technical result is the super-Weyl transformation law of covariant projective supermultiplets [11]. Under the super-Weyl transformation given by eqs. (4.22) and (4.39), which relates the AdS covariant derivatives to the flat ones, the AdS projective supermultiplet can be represented as follows:

$$\mathcal{Q}^{(n)}(v) = \mathrm{e}^{\frac{1}{2}n(\sigma+\bar{\sigma})}Q^{(n)}(v) \ , \tag{5.1}$$

where $Q^{(n)}(v)$ is a projective multiplet in flat superspace,

$$D_\alpha^{(1)}Q^{(n)} = \bar{D}_\alpha^{(1)}Q^{(n)} = 0 \ , \tag{5.2}$$

with the derivatives $D_\alpha^{(1)}$ and $\bar{D}_\alpha^{(1)}$ defined in (4.45). Using eq. (5.1), the action (1.23) can be transformed to a form that involves integration over four Grassmann directions [10], instead of the eight-dimensional Grassmann variables in (1.23).

It follows from eqs. (4.22), (4.30), (4.34) and (5.1) that

$$\mathcal{L}^{(2)} = \mathrm{e}^{\sigma+\bar{\sigma}}L^{(2)} \ , \qquad \mathcal{S}^{(2)} = \mathrm{e}^{\sigma+\bar{\sigma}}\Sigma^{(2)} \ , \qquad \mathbb{E} = 1 \ . \tag{5.3}$$

Then the action (1.23) can be rewritten as

$$\begin{aligned}
S &= \frac{1}{2\pi} \oint_C (v, \mathrm{d}v) \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathrm{d}^4\bar{\theta} \, \frac{\mathrm{e}^{-\sigma-\bar{\sigma}}L^{(2)}}{(\Sigma^{(2)})^2} \\
&= \frac{1}{2\pi} \oint_C (v, \mathrm{d}v) \int \mathrm{d}^4x \, D^{(-4)}D^{(4)} \, \frac{\mathrm{e}^{-\sigma-\bar{\sigma}}L^{(2)}}{(\Sigma^{(2)})^2} \Big|_{\theta=0} \ ,
\end{aligned} \tag{5.4}$$

where we have defined

$$D^{(4)} := \frac{1}{16}D^{(2)}\bar{D}^{(2)} \ , \qquad D^{(2)} := v_i v_j D^{ij} \ , \qquad \bar{D}^{(2)} := v_i v_j \bar{D}^{ij} \ ; \tag{5.5a}$$

$$D^{(-4)} := \frac{1}{16}\frac{u_i u_j u_k u_l}{(v,u)^4}D^{ij}\bar{D}^{kl} \ . \tag{5.5b}$$

Here again u_i is an auxiliary isotwistor which is only subject to the condition $(v, u) := v^i u_i \neq 0$, and is otherwise completely arbitrary. Since $L^{(2)}$ and $\Sigma^{(2)}$ are flat projective supermultiplets, it remains to use the identity

$$D^{(4)}\mathrm{e}^{-\sigma-\bar{\sigma}} = \left(\frac{1}{4}D^{(2)}W\right)\left(\frac{1}{4}\bar{D}^{(2)}\bar{W}\right) = (\Sigma^{(2)})^2 \tag{5.6}$$

to end up with

$$S = \frac{1}{2\pi} \oint_C (v, \mathrm{d}v) \int \mathrm{d}^4x \, D^{(-4)} \, L^{(2)} \Big|_{\theta=0} \ . \tag{5.7}$$

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The new form of the supersymmetric action obtained, eq. (5.7), is much simpler than the original one, eq. (1.23). However, this is not our final representation for the action to work with. There are reasons to look for further simplifications. The point is that we are mostly interested in the off-shell σ-model (1.26). The flat-superspace version of the corresponding Lagrangian is

$$L^{(2)} = \frac{1}{2s}\Sigma^{(2)} K(\Upsilon, \check{\Upsilon}) \ , \tag{5.8}$$

with $\Sigma^{(2)}$ given by (4.49). Upon projection of this Lagrangian to the $\mathcal{N} = 2$ subspace of the 3D $\mathcal{N} = 4$ superspace, $L^{(2)}|_{\theta_2 = \bar{\theta}^2 = 0}$, it turns out that $\Sigma^{(2)}$ has a nontrivial dependence on the isotwistor v^i, and this complicates the evaluation of the contour integral (5.7). This problem can be avoided by choosing an alternative $\mathcal{N} = 2$ subspace of the 3D $\mathcal{N} = 4$ superspace.

5.2 New Grassmann coordinates for 3D $\mathcal{N} = 4$ central charge superspace

In performing reduction from four to three dimensions, the manifest 4D Lorentz symmetry gets broken down to the 3D one, and the 4D $\mathcal{N} = 2$ flat spinor derivatives (4.26) turn into the 3D $\mathcal{N} = 4$ covariant derivatives defined in (4.36). Since the difference between dotted and undotted indices disappears in three dimensions, we are in a position to introduce a new basis for the spinor covariant derivatives defined as

$$\boldsymbol{D}_\alpha^i := \frac{1}{\sqrt{2}}(D_\alpha^i + \lambda \bar{D}_\alpha^i) \ , \qquad \lambda \in \mathbb{C} \ . \tag{5.9a}$$

By complex conjugation we find

$$\bar{\boldsymbol{D}}_{\alpha i} = \frac{1}{\sqrt{2}}(\bar{D}_{\alpha i} - \bar{\lambda} D_{\alpha i}) \ , \qquad \overline{(\boldsymbol{D}_\alpha^i \Psi)} = (-1)^{\varepsilon(\Psi)} \bar{\boldsymbol{D}}_{\alpha i} \bar{\Psi} \ , \tag{5.9b}$$

where $\varepsilon(\Psi)$ denotes the Grassmann parity of a complex superfield Ψ. Using (4.37) we derive the anti-commutation relations obeyed by the operators introduced:

$$\left\{\boldsymbol{D}_\alpha^i, \boldsymbol{D}_\beta^j\right\} = -2\lambda \varepsilon^{ij}\varepsilon_{\alpha\beta}\partial_z \ , \qquad \left\{\bar{\boldsymbol{D}}_{\alpha i}, \bar{\boldsymbol{D}}_{\beta j}\right\} = -2\bar{\lambda}\varepsilon_{ij}\varepsilon_{\alpha\beta}\partial_z \tag{5.10a}$$

$$\left\{\boldsymbol{D}_\alpha^i, \bar{\boldsymbol{D}}_{\beta j}\right\} = -\mathrm{i}(1 + |\lambda|^2)\delta_j^i\partial_{\alpha\beta} + (1 - |\lambda|^2)\delta_j^i\varepsilon_{\alpha\beta}\partial_z \ . \tag{5.10b}$$

Making the choice

$$\lambda\bar{\lambda} = 1 \tag{5.11}$$

simplifies the above algebra

$$\{\boldsymbol{D}_\alpha^i, \boldsymbol{D}_\beta^j\} = -2\lambda\varepsilon^{ij}\varepsilon_{\alpha\beta}\partial_z \ , \qquad \{\bar{\boldsymbol{D}}_{\alpha i}, \bar{\boldsymbol{D}}_{\beta j}\} = -2\bar{\lambda}\varepsilon_{ij}\varepsilon_{\alpha\beta}\partial_z \ , \tag{5.12a}$$

$$\{\boldsymbol{D}_\alpha^i, \bar{\boldsymbol{D}}_{\beta j}\} = -2\mathrm{i}\delta_j^i\partial_{\alpha\beta} \ . \tag{5.12b}$$

An explicit realization of the covariant derivatives introduced is as follows:

$$\boldsymbol{D}_\alpha^i = \frac{\partial}{\partial\boldsymbol{\theta}_i^\alpha} + \mathrm{i}(\gamma^{\hat{m}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^{\beta i}\partial_{\hat{m}} - \lambda\boldsymbol{\theta}_\alpha^i\partial_z \ , \tag{5.13a}$$

$$\bar{\boldsymbol{D}}_{\alpha i} = -\frac{\partial}{\partial\bar{\boldsymbol{\theta}}^{\alpha i}} - \mathrm{i}(\gamma^{\hat{m}})_{\alpha\beta}\boldsymbol{\theta}_i^\beta\partial_{\hat{m}} + \bar{\lambda}\bar{\boldsymbol{\theta}}_{\alpha i}\partial_z \ , \tag{5.13b}$$

where we have defined

$$\boldsymbol{\theta}_i^\alpha := \frac{1}{\sqrt{2}}\left(\theta_i^\alpha + \bar{\lambda}\bar{\theta}_i^\alpha\right) \ , \qquad \bar{\boldsymbol{\theta}}_\alpha^i := \frac{1}{\sqrt{2}}\left(\bar{\theta}_\alpha^i - \lambda\theta_\alpha^i\right) \ . \tag{5.14}$$

Let $Q^{(n)}(v)$ be a weight-n projective multiplet,

$$D_\alpha^{(1)}Q^{(n)} = \bar{D}_\alpha^{(1)}Q^{(n)} = 0 \ , \tag{5.15}$$

with respect to the covariant derivatives (4.45) obeying the anti-commutation relations (4.46). We can extend the definition (4.45) to the case of the covariant derivatives (5.9) by defining

$$\boldsymbol{D}_\alpha^{(1)} := \boldsymbol{D}_\alpha^i v_i \ , \qquad \bar{\boldsymbol{D}}_\alpha^{(1)} := \bar{\boldsymbol{D}}_\alpha^i v_i \ . \tag{5.16}$$

It follows from the definition of $\boldsymbol{D}_\alpha^i$ and $\bar{\boldsymbol{D}}_\alpha^i$ that

$$\boldsymbol{D}_\alpha^{(1)} = \frac{1}{\sqrt{2}}(D_\alpha^{(1)} + \lambda\bar{D}_\alpha^{(1)}) \ , \qquad \bar{\boldsymbol{D}}_\alpha^{(1)} = \frac{1}{\sqrt{2}}(\bar{D}_\alpha^{(1)} - \bar{\lambda}D_\alpha^{(1)}) \ . \tag{5.17}$$

In other words, the operators $\boldsymbol{D}_\alpha^{(1)}$ and $\bar{\boldsymbol{D}}_\alpha^{(1)}$ are related to $D_\alpha^{(1)}$ and $\bar{D}_\alpha^{(1)}$ by a linear unimodular transformation. Therefore, if $Q^{(n)}$ is a projective supermultiplet with respect to the covariant derivatives (4.45), it is also a projective supermultiplet with respect to the spinor derivatives $\boldsymbol{D}_\alpha^{(1)}$ and $\bar{\boldsymbol{D}}_\alpha^{(1)}$,

$$\boldsymbol{D}_\alpha^{(1)}Q^{(n)} = \bar{\boldsymbol{D}}_\alpha^{(1)}Q^{(n)} = 0 \ , \tag{5.18}$$

and vice versa. We conclude that the two sets of operators, $(D_\alpha^{(1)}, \bar{D}_\alpha^{(1)})$ and $(\boldsymbol{D}_\alpha^{(1)}, \bar{\boldsymbol{D}}_\alpha^{(1)})$, are completely equivalent to use when dealing with the projective supermultiplets.

There are two technical reasons why the use of covariant derivatives (5.9), and the associated Grassmann variables (5.14), are advantageous in the context of supersymmetric

σ-models in projective superspace. First of all, it is this realization of 3D $\mathcal{N} = 4$ central charge superspace which provides the simplest embedding of 3D $\mathcal{N} = 2$ superspace *without central charge*. Given an arbitrary $\mathcal{N} = 4$ superfield $U(\boldsymbol{\theta}_i^\alpha, \boldsymbol{\theta}_\alpha^i)$, we define its $\mathcal{N} = 2$ projection as follows

$$U| := U\Big|_{\boldsymbol{\theta}_{\underline{2}} = \boldsymbol{\theta}^{\underline{2}} = 0} . \tag{5.19}$$

Defining also the Grassmann variables of $\mathcal{N} = 2$ superspace,

$$\boldsymbol{\theta}^\alpha := \boldsymbol{\theta}_{\underline{1}}^\alpha = \frac{1}{\sqrt{2}}(\theta_{\underline{1}}^\alpha - \bar{\lambda}\bar{\theta}^{\alpha\underline{2}}) , \qquad \bar{\boldsymbol{\theta}}_\alpha := \bar{\boldsymbol{\theta}}_\alpha^{\underline{1}} = \frac{1}{\sqrt{2}}(\bar{\theta}_\alpha^{\underline{1}} - \lambda\theta_{\alpha\underline{2}}) , \tag{5.20}$$

it is easy to see that

$$\boldsymbol{D}_\alpha := \boldsymbol{D}_\alpha^{\underline{1}}\Big|_{\boldsymbol{\theta}_{\underline{2}} = 0} = \frac{\partial}{\partial\boldsymbol{\theta}^\alpha} + \mathrm{i}(\gamma^{\hat{m}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^\beta\partial_{\hat{m}} , \tag{5.21a}$$

$$\bar{\boldsymbol{D}}_\alpha := \bar{\boldsymbol{D}}_{\alpha\underline{1}}\Big|_{\boldsymbol{\theta}_{\underline{2}} = 0} = -\frac{\partial}{\partial\bar{\boldsymbol{\theta}}^\alpha} - \mathrm{i}(\gamma^{\hat{m}})_{\alpha\beta}\boldsymbol{\theta}^\beta\partial_{\hat{m}} . \tag{5.21b}$$

These operators have no dependence on z, and obey the anti-commutation relations

$$\{\boldsymbol{D}_\alpha, \boldsymbol{D}_\beta\} = \{\bar{\boldsymbol{D}}_\alpha, \bar{\boldsymbol{D}}_\beta\} = 0 , \qquad \{\boldsymbol{D}_\alpha, \bar{\boldsymbol{D}}_\beta\} = -2\mathrm{i}(\gamma^{\hat{m}})_{\alpha\beta}\partial_{\hat{m}} \tag{5.22}$$

corresponding to 3D $\mathcal{N} = 2$ Poincaré supersymmetry without central charge. Note that the similar $\theta^{\underline{2}} = \bar{\theta}_{\underline{2}} = 0$ reduction of the derivatives $(D_\alpha^{\underline{1}}, \bar{D}_{\alpha\underline{1}})$, eq. (4.36), does not decouple the z-dependence, leading to the 3D $\mathcal{N} = 2$ superspace with central charge.

The second reason to use the covariant derivatives (5.9) is that this realization turns out to provide, for a special value of λ, the simplest expression for the $\mathcal{N} = 2$ projection of $\Sigma^{[2]}$. Let us look again at $\Sigma^{(2)}(v)$. If we restrict our attention to the open domain of $\mathbb{C}P^1$ where $v^{\underline{1}} \neq 0$, which is the north chart of $\mathbb{C}P^1$, we can write

$$\Sigma^{(2)}(v) = \mathrm{i}(v^{\underline{1}})^2\zeta\Sigma^{[2]}(\zeta) , \qquad \Sigma^{[2]}(\zeta) = -\frac{\mathrm{i}}{\zeta}\Sigma^{ij}\zeta_i\zeta_j , \tag{5.23}$$

where

$$v^i = v^{\underline{1}}(1, \zeta) = v^{\underline{1}}\zeta^i , \quad \zeta^i = (1, \zeta) , \quad \zeta_i = (-\zeta, 1) . \tag{5.24}$$

Now, we consider the case in which only one of the three independent components of $\boldsymbol{s}^{ij}$ is nonzero

$$s^{\underline{11}} = s^{\underline{22}} = 0 , \quad s^{\underline{12}} = \alpha s , \qquad \alpha = \pm\mathrm{i} , \tag{5.25}$$

which is exactly the choice (1.3a). Note that s is real and positive. A straightforward computation of the 3D $\mathcal{N} = 2$ reduction of $\Sigma^{[2]}(\zeta)$ gives

$$
\begin{aligned}
\Sigma^{[2]}(\zeta)\big| = {} & 2\mathrm{i}\alpha s^{-1}z^{-2} \\
& -\mathrm{i}s^{-1}z^{-3}\Big(2(\lambda+\bar{\lambda})\boldsymbol{\theta}^\alpha\bar{\boldsymbol{\theta}}_\alpha + \frac{1}{\zeta}(\lambda^2-1+2\alpha\lambda)\boldsymbol{\theta}^2 + \zeta(\bar{\lambda}^2-1-2\alpha\bar{\lambda})\bar{\boldsymbol{\theta}}^2\Big) \\
& -3\mathrm{i}s^{-1}z^{-4}(\lambda-\bar{\lambda}+2\alpha)\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}}^2 \ .
\end{aligned}
\tag{5.26}
$$

So far, the parameter λ has been restricted by $|\lambda| = 1$. We now observe that choosing

$$
\lambda = -\alpha = \mp\mathrm{i} \ ,
\tag{5.27}
$$

allows us to eliminate all the $\boldsymbol{\theta}$ and ζ dependence of $\Sigma_0^{(2)}\big|$ (in complete analogy to the 5D case [16]). The condition (5.27) completely fixes the 3D $\mathcal{N} = 2$ reduction to be used below.

Note that the choice of sign in (5.27) is conventional. In fact one can flip the sign by changing everywhere the $\boldsymbol{\theta}$ coordinates to $\bar{\boldsymbol{\theta}}$ and vice versa. Throughout the main body of the paper we will use $\lambda = -\alpha = \mathrm{i}$, which simplifies (5.26) to

$$
\Sigma^{[2]}\big| = 2s^{-1}z^{-2} \ .
\tag{5.28}
$$

5.3   The supersymmetric action revisited

We now return to the supersymmetric action (5.7). It is given in terms of the flat covariant derivatives $(D_\alpha^i, \bar{D}_\alpha^i)$. Above we have introduced the new basis for the spinor covariant derivatives, $(\boldsymbol{D}_\alpha^i, \bar{\boldsymbol{D}}_\alpha^i)$. We have also shown that any projective supermultiplet with respect to $(D_\alpha^i, \bar{D}_\alpha^i)$ is also projective with respect to $(\boldsymbol{D}_\alpha^i, \bar{\boldsymbol{D}}_\alpha^i)$. In terms of the new spinor derivatives, the action can be seen to be

$$
S = \frac{1}{2\pi} \oint_C (v, \mathrm{d}v) \int \mathrm{d}^3x \, \mathrm{d}z \, \boldsymbol{D}^{(-4)} L^{(2)}\Big|_{\theta=0} \ ,
\tag{5.29}
$$

where

$$
\boldsymbol{D}^{(-4)} := \frac{1}{16(v, u)^4} u_i u_j u_k u_l \boldsymbol{D}^{ij} \bar{\boldsymbol{D}}^{kl} \ .
\tag{5.30}
$$

Without loss of generality, we can assume the north pole of $\mathbb{C}P^1$, $v^i \propto (0, 1)$, to lie outside of the integration contour in (5.7), and hence we introduce the complex inhomogeneous coordinate ζ for $\mathbb{C}P^1$ defined by $v^i = v^{\underline{1}}(1, \zeta)$. Since the action (5.7) is independent

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of u_i, we can also choose $u_i = (1,0)$. In addition, it is standard to represent $L^{(2)}$ in the form

$$L^{(2)}(v) = \mathrm{i}\, v^{\underline{1}} v^{\underline{2}}\, L(\zeta) = \mathrm{i}\big(v^{\underline{1}}\big)^2 \zeta\, L(\zeta)\ . \tag{5.31}$$

By using the fact that $L^{(2)}$ enjoys the constraints $\zeta_i \boldsymbol{D}_\alpha^i L = \zeta_i \bar{\boldsymbol{D}}_\alpha^i L = 0$, we can finally rewrite S as an integral over the 3D $\mathcal{N} = 2$ superspace followed by an integral over $\mathbb{R}_+$

$$S = \int \mathrm{d}z \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}}\, L(\zeta)\Big|_{\boldsymbol{\theta}_{\underline{2}}=\bar{\boldsymbol{\theta}}^{\underline{2}}=0}\ . \tag{5.32}$$

This action is manifestly invariant under the 3D $\mathcal{N} = 2$ super-Poicaré group without central charge. By construction, the action is in fact invariant under the larger 4D $\mathcal{N} = 2$ AdS supergroup $\mathrm{OSp}(2|4)$.

Using our earlier result (5.28), for the supersymmetric σ-model (5.8) we end up with the following action:

$$S = \int \frac{\mathrm{d}z}{(sz)^2} \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}}\, K(\Upsilon^I, \breve{\Upsilon}^{\bar{I}})\Big|_{\boldsymbol{\theta}_{\underline{2}}=\bar{\boldsymbol{\theta}}^{\underline{2}}=0}\ . \tag{5.33}$$

The arctic $\Upsilon^I(\zeta)$ and antarctic $\breve{\Upsilon}^{\bar{I}}(\zeta)$ dynamical variables are generated by an infinite set of ordinary 3D $\mathcal{N} = 2$ superfields parametrically depending on z:

$$\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n\, \Upsilon_n^I = \Phi^I + \zeta\, \Sigma^I + O(\zeta^2)\ , \tag{5.34a}$$

$$\breve{\Upsilon}^{\bar{I}}(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n}\, \bar{\Upsilon}_n^{\bar{I}} = \bar{\Phi}^{\bar{I}} - \frac{1}{\zeta}\bar{\Sigma}^{\bar{I}} + O(\zeta^{-2})\ . \tag{5.34b}$$

Here the physical superfields $\Phi^I := \Upsilon_0^I$ and $\Sigma^I := \Upsilon_1^I$ are 3D $\mathcal{N} = 2$ chiral and complex linear respectively,

$$\bar{\boldsymbol{D}}_\alpha \Phi^I = 0\ , \qquad \bar{\boldsymbol{D}}^2 \Sigma^I = 0\ , \tag{5.35}$$

while the remaining components, $\Upsilon_2^I, \Upsilon_3^I, \ldots$, are complex unconstrained 3D $\mathcal{N} = 2$ superfields. The crucial point is that, except for the overall integral $\int \mathrm{d}z\,(sz)^{-2}$, the action (5.33) looks exactly like a flat 3D $\mathcal{N} = 4$ σ-model, see e.g. [52].

6 σ-models from projective superspace: 3D foliated frame

Our next task is to reformulate the AdS supersymmetric σ-model (5.33) in terms of 3D $\mathcal{N} = 2$ chiral superfields by formally eliminating the auxiliary superfields $\Upsilon_2^I, \Upsilon_3^I, \ldots$,

and performing an appropriate duality transformation. In this and subsequent sections, we will be dealing only with 3D $\mathcal{N} = 2$ superfields (parametrically depending on the forth space variable z), so we will always drop the explicit bar-projection.

The action (5.33) possesses off-shell 4D $\mathcal{N} = 2$ AdS supersymmetry, $\text{OSp}(2|4)$, by construction. Part of this supersymmetry is manifest as 3D $\mathcal{N} = 2$ Poincaré supersymmetry. To exhibit explicitly the remaining symmetries, we need to know the structure of the Killing vector fields in this foliated superspace. Since the generic form of the Killing vector fields in AdS$^{4|8}$ is known [10], it is a straightforward task to specialize them to the specific frame we have chosen. We leave the details to appendix A and give only the result here when projected to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^{\underline{2}} = 0$.

6.1 Extended supersymmetry and Killing vector fields

Within the 3D foliated frame of AdS, a projective supermultiplet $\mathcal{Q}(\zeta)$ of weight zero transforms as

$$\delta\mathcal{Q} = -\left(\xi^{\hat{a}}\partial_{\hat{a}} + \xi^z\partial_z + \xi^\alpha \boldsymbol{D}_\alpha + \bar{\xi}_\alpha \bar{\boldsymbol{D}}^\alpha + \zeta\rho^\alpha \boldsymbol{D}_\alpha - \frac{1}{\zeta}\bar{\rho}_\alpha \bar{\boldsymbol{D}}^\alpha - 2\mathrm{i}\Lambda\zeta\partial_\zeta\right)\mathcal{Q} \, . \qquad (6.1)$$

The parameters $\xi^{\hat{a}}$, ξ^z, etc., can be decomposed into those associated with the manifest 3D $\mathcal{N} = 2$ Poincaré supersymmetry

$$\xi^{\hat{a}} = p^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}}x^{\hat{b}} - \frac{\mathrm{i}}{2}\varepsilon^{\hat{a}\hat{b}\hat{c}}\omega_{\hat{b}\hat{c}}\boldsymbol{\theta}^\alpha\bar{\boldsymbol{\theta}}_\alpha - 2\mathrm{i}\,\bar{\epsilon}_\alpha(\gamma^{\hat{a}})^{\alpha\beta}\boldsymbol{\theta}_\beta - 2\mathrm{i}\,\epsilon^\alpha(\gamma^{\hat{a}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^\beta \, , \qquad (6.2a)$$

$$\xi^\alpha = \epsilon^\alpha - \frac{1}{4}\omega^{\hat{b}\hat{c}}\varepsilon_{\hat{b}\hat{c}\hat{d}}\boldsymbol{\theta}^\beta(\gamma^{\hat{d}})_\beta{}^\alpha \, , \qquad (6.2b)$$

$$\xi^z = \rho^\alpha = \bar{\rho}_\alpha = \Lambda = 0 \, , \qquad (6.2c)$$

and those associated with the remaining isometries

$$\xi^{\hat{a}} = 2rx^{\hat{a}} + 2\mathrm{i}\Lambda_{\underline{1}}{}^{\underline{1}}\boldsymbol{\theta}^\alpha(\gamma^{\hat{a}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^\beta - 2x^{\hat{b}}k_{\hat{b}}x^{\hat{a}} + x^2k^{\hat{a}} - 2\mathrm{i}\varepsilon^{\hat{a}\hat{b}\hat{c}}k_{\hat{b}}x_{\hat{c}}\boldsymbol{\theta}^\alpha\bar{\boldsymbol{\theta}}_\alpha + z^2k^{\hat{a}}$$
$$- \frac{1}{2}\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}}^2k^{\hat{a}} + 2x^{\hat{a}}\eta^\alpha\boldsymbol{\theta}_\alpha + 2x^{\hat{a}}\bar{\eta}_\alpha\bar{\boldsymbol{\theta}}^\alpha - 2x^{\hat{b}}\varepsilon^{\hat{a}}{}_{\hat{b}\hat{c}}\eta^\alpha(\gamma^{\hat{c}})_\alpha{}^\beta\boldsymbol{\theta}_\beta + 2x^{\hat{b}}\varepsilon^{\hat{a}}{}_{\hat{b}\hat{c}}\bar{\eta}_\alpha(\gamma^{\hat{c}})^\alpha{}_\beta\bar{\boldsymbol{\theta}}^\beta$$
$$- \mathrm{i}\,\eta^\alpha(\gamma^{\hat{a}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^\beta\boldsymbol{\theta}^2 - \mathrm{i}\,\bar{\eta}^\alpha(\gamma^{\hat{a}})_{\alpha\beta}\boldsymbol{\theta}^\beta\bar{\boldsymbol{\theta}}^2 \, , \qquad (6.3a)$$

$$\xi^\alpha = -\Lambda_{\underline{1}}{}^{\underline{1}}\boldsymbol{\theta}^\alpha - k_{\hat{b}}x^{\hat{b}}\boldsymbol{\theta}^\alpha - k^{\hat{b}}x^{\hat{c}}\varepsilon_{\hat{b}\hat{c}\hat{d}}\boldsymbol{\theta}^\beta(\gamma^{\hat{d}})_\beta{}^\alpha - \frac{\mathrm{i}}{2}k_{\hat{b}}\boldsymbol{\theta}^2\bar{\boldsymbol{\theta}}^\beta(\gamma^{\hat{b}})_\beta{}^\alpha - \mathrm{i}x^{\hat{b}}\bar{\boldsymbol{\eta}}_\beta(\gamma_{\hat{b}})^{\beta\alpha}$$
$$+ \boldsymbol{\theta}^\beta\bar{\boldsymbol{\theta}}_\beta\bar{\eta}^\alpha + \eta^\alpha\boldsymbol{\theta}^2 + 2\boldsymbol{\theta}^\beta\bar{\eta}_\beta\bar{\boldsymbol{\theta}}^\alpha \, , \qquad (6.3b)$$

$$\xi^z = 2rz - 2zx^{\hat{b}}k_{\hat{b}} + 2z\eta^\alpha\boldsymbol{\theta}_\alpha + 2z\bar{\eta}_\alpha\bar{\boldsymbol{\theta}}^\alpha \, , \qquad (6.3c)$$

$$\rho^\alpha = \mathrm{i}z\eta^\alpha + zk_{\hat{b}}\bar{\boldsymbol{\theta}}^\beta(\gamma^{\hat{b}})_\beta{}^\alpha \, , \qquad (6.3d)$$

$$\Lambda = -\mathrm{i}\Lambda_{\underline{1}}{}^{\underline{1}} - 2\mathrm{i}\eta^\alpha\boldsymbol{\theta}_\alpha + 2\mathrm{i}\bar{\eta}_\alpha\bar{\boldsymbol{\theta}}^\alpha - 2k_{\hat{a}}\boldsymbol{\theta}^\alpha(\gamma^{\hat{a}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^\beta \, . \qquad (6.3e)$$

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A general isometry is the combination of the two given. From now on, we will always consider the combination.

Several observations should be made. The isometry transformations (6.2) can be easily identified with those of 3D $\mathcal{N} = 2$ Minkowski superspace: the constant parameters $p^{\hat{a}}$, $\omega_{\hat{a}\hat{b}}$, and ϵ^{α} correspond exactly to the 3D translations, Lorentz rotations, and supersymmetry. In contrast, the isometries (6.3) are more complicated. The parameter r can be identified with translations in the z direction, η^{α} parametrizes the additional supersymmetry, and $\Lambda_{\underline{1}}{}^{\underline{1}}$ is plainly the SO(2) R-symmetry. The parameter $k_{\hat{a}}$ may be interpreted as a remnant of the four-dimensional Lorentz symmetry.

An alternative interpretation of these isometries is suggested by the well-known fact that the 4D $\mathcal{N} = 2$ AdS supergroup and the 3D $\mathcal{N} = 2$ superconformal group are isomorphic to the supergroup OSp(2|4). From the 3D superconformal point of view, the constant parameters r, η^{α}, $\Lambda_{\underline{1}}{}^{\underline{1}}$, and $k_{\hat{a}}$ are associated with the 3D dilatations, S-supersymmetry, R-symmetry, and special conformal transformations, respectively. The above set of transformations is a certain realization of the 3D $\mathcal{N} = 2$ superconformal group. Observe that

$$\bar{\boldsymbol{D}}^{\beta}\xi^{\alpha} = 0 \;, \qquad \bar{\boldsymbol{D}}^{\beta}\xi^{\hat{a}} = 2\mathrm{i}\xi_{\gamma}(\gamma^{\hat{a}})^{\gamma\beta} \;, \qquad \bar{\boldsymbol{D}}^{(\gamma}\xi^{\beta\alpha)} = 0 \;, \tag{6.4}$$

which imply that $\xi^{\hat{a}}$ and ξ^{α} are respectively the vector and spinor components of a 3D $\mathcal{N} = 2$ *superconformal* Killing vector field [52]. In fact, except for the $z^2 k^{\hat{a}}$ term in $\xi^{\hat{a}}$, the general expression for $\xi^{\hat{a}}$ is the most general solution of these constraints in 3D $\mathcal{N} = 2$ superspace. From the 3D point of view, the $z^2 k^{\hat{a}}$ term is just a constant three-vector which can be combined with $p^{\hat{a}}$ into a constant parameter of translations.

The SO(2) superfield parameter Λ in (6.1) obeys

$$\Lambda = \frac{\mathrm{i}}{4}(\boldsymbol{D}_{\alpha}\xi^{\alpha} - \bar{\boldsymbol{D}}^{\alpha}\bar{\xi}_{\alpha}) \tag{6.5}$$

which identifies it as the SO(2) superfield parameter of the 3D superconformal group. The 3D superconformal scale parameter σ may be identified as

$$\sigma = \frac{1}{2}(\boldsymbol{D}_{\alpha}\xi^{\alpha} + \bar{\boldsymbol{D}}^{\alpha}\bar{\xi}_{\alpha}) = \frac{1}{3}\partial_{\hat{a}}\xi^{\hat{a}} = 2r - 2x^{\hat{a}}k_{\hat{a}} + 2\eta^{\alpha}\boldsymbol{\theta}_{\alpha} + 2\bar{\eta}_{\alpha}\bar{\boldsymbol{\theta}}^{\alpha} \;, \tag{6.6}$$

and a certain combination of σ and Λ must be chiral,

$$\boldsymbol{\sigma} := \sigma + \mathrm{i}\Lambda = 2r + \Lambda_{\underline{1}}{}^{\underline{1}} + 4\eta^{\alpha}\boldsymbol{\theta}_{\alpha} - 2k_{\hat{b}}(x^{\hat{b}} + \mathrm{i}\boldsymbol{\theta}^{\alpha}(\gamma^{\hat{b}})_{\alpha\beta}\bar{\boldsymbol{\theta}}^{\beta}) \;, \qquad \bar{\boldsymbol{D}}^{\alpha}\boldsymbol{\sigma} = 0 \;. \tag{6.7}$$

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The remaining parameters in (6.1) obey

$$\xi^z = z\sigma \ , \qquad \rho^\alpha = \frac{iz}{2} \boldsymbol{D}^\alpha \sigma \ , \qquad \boldsymbol{D}_\beta \rho^\alpha = \bar{\boldsymbol{D}}^\beta \bar{\rho}_\alpha = 0 \ . \tag{6.8}$$

All of the parameters discussed above can be derived from $\xi^{\hat{a}}$ using the equations given. There is one difference between the parameter $\xi^{\hat{a}}$ and the usual 3D $\mathcal{N} = 2$ superconformal parameter $\xi^{\hat{a}}$: the explicit z dependence in the $z^2 k^{\hat{a}}$ term. This leads to the additional identity

$$\partial_z \xi_{\hat{a}} = \partial_{\hat{a}} \xi^z = z \partial_{\hat{a}} \sigma \ . \tag{6.9}$$

Note that ξ^α and σ possess no z-dependence.

It is straightforward to work out the algebra of the isometry transformations. Defining $\delta_{21} = [\delta_2, \delta_1]$, one finds that

$$\xi_{21}^{\hat{a}} = 2\xi_{[2}^B \boldsymbol{D}_B \xi_{1]}^{\hat{a}} + 4i\, \xi_{[2}^\alpha \bar{\xi}_{1]}^\beta (\gamma^{\hat{a}})_{\alpha\beta} + 2\xi_{[2}^z \partial_z \xi_{1]}^{\hat{a}} + 4i\, \rho_{[2}^\alpha \bar{\rho}_{1]}^\beta (\gamma^{\hat{a}})_{\alpha\beta} \ . \tag{6.10}$$

From this parameter, one can derive the rest. Note that this differs from the usual 3D $\mathcal{N} = 2$ superconformal algebra in the presence of the third and fourth terms, both of which are proportional to z^2. These are consistent with the presence of the $z^2 k^{\hat{a}}$ term in $\xi^{\hat{a}}$. Although the specific transformations of the Killing vectors have been modified, the transformation induced on the constant parameters $p^{\hat{a}}$, $\omega^{\hat{a}}{}_{\hat{b}}$, ϵ, r, $\Lambda_{\underline{1}}{}^{\underline{1}}$, $k_{\hat{a}}$, η^α remains that of the supergroup OSp(2|4).

The above identities allow us to easily check that the action (5.33) is invariant. Since the Lagrangian K is a weight-zero projective multiplet, the variation of the action is

$$\delta S = -\int \frac{\mathrm{d}z}{(sz)^2} \oint_C \frac{\mathrm{d}\zeta}{2\pi i\, \zeta} \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}} \left(\xi^{\hat{a}} \partial_{\hat{a}} K + (\xi^\alpha + \zeta\rho^\alpha)\boldsymbol{D}_\alpha K + (\bar{\xi}_\alpha - \frac{1}{\zeta}\bar{\rho}_\alpha)\bar{\boldsymbol{D}}^\alpha K \right.$$
$$\left. + \xi^z \partial_z K - 2i\Lambda\zeta\partial_\zeta K \right) \ . \tag{6.11}$$

The last term is a total contour derivative and vanishes. The first three terms can be integrated by parts to give

$$\delta S = \oint_C \frac{\mathrm{d}\zeta}{2\pi i\zeta} \int \frac{\mathrm{d}z}{(sz)^2} \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}} \left((\partial_{\hat{a}}\xi^{\hat{a}} - \boldsymbol{D}_\alpha\xi^\alpha - \bar{\boldsymbol{D}}^\alpha\bar{\xi}_\alpha)K - \xi^z\partial_z K \right)$$
$$= \oint_C \frac{\mathrm{d}\zeta}{2\pi i\zeta} \int \frac{\mathrm{d}z}{(sz)^2} \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}} \left(\sigma K - z\sigma\partial_z K \right)$$
$$= -\oint_C \frac{\mathrm{d}\zeta}{2\pi i\zeta} \int \mathrm{d}z \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}}\, \partial_z\left(\frac{\sigma K}{s^2 z} \right) \ , \tag{6.12}$$

which vanishes provided we can dispense with total derivatives in z.[16]

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[16]When analyzing invariance of the action under infinitesimal isometry transformations, it suffices to restrict the fields to be supported inside the Poincaré patch.

6.2   Elimination of auxiliaries

We emphasize that the σ-model action (5.33) is a *four-dimensional* action involving hypermultiplets with an infinite number of auxiliary fields. Upon elimination of the auxiliaries and a duality transformation, this action is naturally associated with a four-dimensional $\mathcal{N} = 2$ supersymmetric σ-model in AdS. Our peculiar parametrization of $\mathrm{AdS}^{4|8}$ has been chosen to render this elimination of auxiliaries as simple as possible. Let us now turn to this task.

The elimination of auxiliaries begins with the following observation. Performing the contour integral yields

$$\oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}) = \mathcal{L}(\Upsilon_n^I, \bar{\Upsilon}_n^{\bar{J}}) \tag{6.13}$$

where $\mathcal{L}$ is some function depending on all of the components Υ_n^I of the arctic multiplets and their conjugates. For $n \geq 2$, these components are unconstrained from the point of view of 3D $\mathcal{N} = 2$ superspace. Putting them on-shell naturally leads to an infinite set of nonlinear algebraic equations

$$0 = \frac{\partial \mathcal{L}}{\partial \Upsilon_n^I} = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \frac{\partial K}{\partial \Upsilon^I} \zeta^n \,, \qquad n \geq 2 \tag{6.14a}$$

$$0 = \frac{\partial \mathcal{L}}{\partial \bar{\Upsilon}_n^{\bar{J}}} = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \frac{\partial K}{\partial \check{\Upsilon}^{\bar{J}}} (-\zeta)^{-n} \,, \qquad n \geq 2 \tag{6.14b}$$

which are difficult to solve in general.

However (and this is the point) the equations (6.14) are *exactly* the same as those originating in the $\mathcal{N} = 2$ supersymmetric σ-model in four-dimensional Minkowski space

$$S = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}) \,, \tag{6.15}$$

which was first studied in [18, 19].[17] This naturally allows us to appropriate with little modification the formal technique of reformulating (6.15) in terms of $\mathcal{N} = 1$ chiral superfields [28] (see also [29]) and devote it to solving the problem in AdS. The price we pay is the loss of manifest four-dimensional Lorentz invariance.

Assuming that the equations (6.14) have been satisfied for some choice of the auxiliaries, the action is reduced to

$$S = \int \frac{\mathrm{d}z}{(sz)^2} \int \mathrm{d}^3x \, \mathrm{d}^2\boldsymbol{\theta} \, \mathrm{d}^2\bar{\boldsymbol{\theta}} \, \mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \,, \tag{6.16}$$

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[17]The most general $\mathcal{N} = 2$ supersymmetric σ-model in four-dimensional projective superspace [8] is obtained from (6.15) by allowing the superfield Lagrangian to possess an arbitrary ζ-dependence, $K(\Upsilon, \check{\Upsilon}) \to \boldsymbol{K}(\Upsilon, \check{\Upsilon}, \zeta)$.

where

$$\mathcal{L} := \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \, K(\Upsilon, \check{\Upsilon}) \, . \tag{6.17}$$

Here $\Phi^I = \Upsilon_0^I$ and $\Sigma^I = \Upsilon_1^I$ are the lowest two components of the arctic multiplet Υ^I. These are constrained as

$$\bar{\boldsymbol{D}}^\alpha \Phi^I = 0 \, , \qquad -\frac{1}{4}\bar{\boldsymbol{D}}^2 \Sigma^I = \mathrm{i}\,\partial_z \Phi^I \, . \tag{6.18}$$

In other words, Φ^I is chiral. The condition Σ^I obeys is a modified version of the complex linear condition, so we will refer to Σ^I as a modified complex linear superfield.[18]

The transformation law (6.1) for Υ^I implies transformations for Φ^I and Σ^I:

$$\delta\Phi^I = -\xi^{\hat{a}}\partial_{\hat{a}}\Phi^I - \xi^\alpha \boldsymbol{D}_\alpha \Phi^I + \bar{\rho}_\alpha \bar{\boldsymbol{D}}^\alpha \Sigma^I - \xi^z \partial_z \Phi^I \, , \tag{6.19a}$$

$$\begin{aligned}
\delta\Sigma^I = {} & -\xi^{\hat{a}}\partial_{\hat{a}}\Sigma^I - \xi^\alpha \boldsymbol{D}_\alpha \Sigma^I - \bar{\xi}_\alpha \bar{\boldsymbol{D}}^\alpha \Sigma^I - \rho^\alpha \boldsymbol{D}_\alpha \Phi^I \\
& + \bar{\rho}_\alpha \bar{\boldsymbol{D}}^\alpha \Upsilon_2^I - \xi^z \partial_z \Sigma^I + 2\mathrm{i}\Lambda\Sigma^I \, ,
\end{aligned} \tag{6.19b}$$

where $\Upsilon_2^I = \Upsilon_2^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$. It is straightforward to check that these transformation laws respect the constraints (6.18). In order for $\mathcal{L}$ to be invariant, it must obey a number of constraints which are derivable from its contour definition (6.17). Defining

$$\Xi := \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \frac{K}{\zeta} \, , \tag{6.20}$$

it can be shown that the following conditions hold identically when the auxiliaries have been eliminated:

$$\frac{\partial\mathcal{L}}{\partial\Phi^I} + \frac{\partial\mathcal{L}}{\partial\Sigma^J}\frac{\partial\Upsilon_2^J}{\partial\Sigma^I} = \frac{\partial\Xi}{\partial\Sigma^I} \, , \tag{6.21a}$$

$$-\frac{\partial\mathcal{L}}{\partial\bar{\Sigma}^{\bar{I}}} + \frac{\partial\mathcal{L}}{\partial\Sigma^J}\frac{\partial\Upsilon_2^J}{\partial\bar{\Phi}^{\bar{I}}} = \frac{\partial\Xi}{\partial\bar{\Phi}^{\bar{I}}} \, , \tag{6.21b}$$

$$\frac{\partial\mathcal{L}}{\partial\Sigma^J}\frac{\partial\Upsilon_2^J}{\partial\bar{\Sigma}^{\bar{I}}} = \frac{\partial\Xi}{\partial\bar{\Sigma}^{\bar{I}}} \, . \tag{6.21c}$$

They may be proven using the contour definitions of $\mathcal{L}$ and Ξ. In addition, since K lacks any explicit ζ dependence, one can show [18, 19] that

$$\Sigma^I \frac{\partial\mathcal{L}}{\partial\Sigma^I} = \bar{\Sigma}^{\bar{J}} \frac{\partial\mathcal{L}}{\partial\bar{\Sigma}^{\bar{J}}} \, . \tag{6.22}$$

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[18]Similar modifications to the complex linearity condition are standard in 5D [56] and 6D [57].

Alternatively, requiring that a Lagrangian $\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma})$ be invariant under (6.19), for some unknown function $\Upsilon_2^I = \Upsilon_2^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma})$, leads to a set of equations which imply the existence of some function Ξ obeying the equations (6.21) [28, 29]. Making use of the equations (6.21) and (6.22) leads to the following relation [28]:

$$\Xi = \Sigma^I \frac{\partial \mathcal{L}}{\partial \Phi^I} + 2\Upsilon_2^I \frac{\partial \mathcal{L}}{\partial \Sigma^I} \ . \tag{6.23}$$

This result also follows from the contour integral representation (6.20). Eq. (6.23) is the 3D foliated version of the condition (3.16) which originates in the AdS frame.

Our next task is to perform a duality transformation converting the complex linear variables to a set of purely chiral variables. For this, one relaxes Σ^I to an unconstrained superfield and introduces a Lagrange multiplier chiral superfield Ψ_I, $\bar{D}^\alpha \Psi_I = 0$, and the first-order action

$$S_{\text{F.O.}} = \int \frac{\mathrm{d}z}{(sz)^2} \left\{ \int \mathrm{d}^3 x \, \mathrm{d}^2 \boldsymbol{\theta} \, \mathrm{d}^2 \bar{\boldsymbol{\theta}} \, \left(\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) + \Sigma^I \Psi_I + \bar{\Sigma}^{\bar{I}} \bar{\Psi}_{\bar{I}} \right) \right.$$
$$\left. - \mathrm{i} \int \mathrm{d}^3 x \, \mathrm{d}^2 \boldsymbol{\theta} \, \Psi_I \partial_z \Phi^I + \mathrm{i} \int \mathrm{d}^3 x \, \mathrm{d}^2 \bar{\boldsymbol{\theta}} \, \bar{\Psi}_{\bar{I}} \partial_z \bar{\Phi}^{\bar{I}} \right\} \ . \tag{6.24}$$

Varying $S_{\text{F.O.}}$ with respect to Ψ_I leads to the constraint on Σ^I as in (6.18), and then $S_{\text{F.O.}}$ reduces to the original action, eq. (6.16). Instead, making use of the equation of motion for Σ^I,

$$\frac{\partial \mathcal{L}}{\partial \Sigma^I} = -\Psi_I \ , \tag{6.25}$$

leads to the dual action

$$S_{\text{dual}} = \int \frac{\mathrm{d}z}{(sz)^2} \left\{ \int \mathrm{d}^3 x \, \mathrm{d}^2 \boldsymbol{\theta} \, \mathrm{d}^2 \bar{\boldsymbol{\theta}} \, \mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) \right.$$
$$\left. - \mathrm{i} \int \mathrm{d}^3 x \, \mathrm{d}^2 \boldsymbol{\theta} \, \Psi_I \partial_z \Phi^I + \mathrm{i} \int \mathrm{d}^3 x \, \mathrm{d}^2 \bar{\boldsymbol{\theta}} \, \bar{\Psi}_{\bar{I}} \partial_z \bar{\Phi}^{\bar{I}} \right\} \ , \tag{6.26}$$

where

$$\mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) := \mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) + \Sigma^I \Psi_I + \bar{\Sigma}^{\bar{I}} \bar{\Psi}_{\bar{I}} \tag{6.27}$$

with Σ understood to obey its equation of motion (6.25). The dual action is invariant under the AdS transformations

$$\delta \Phi^I = -\xi^{\hat{a}} \partial_{\hat{a}} \Phi^I - \xi^\alpha \boldsymbol{D}_\alpha \Phi^I - \frac{\mathrm{i}}{4} \bar{\boldsymbol{D}}^2 \left(z\sigma \frac{\partial \mathbb{K}}{\partial \Psi_I} \right) \ , \tag{6.28a}$$

$$\delta \Psi_I = -\xi^{\hat{a}} \partial_{\hat{a}} \Psi_I - \xi^\alpha \boldsymbol{D}_\alpha \Psi_I - 2\boldsymbol{\sigma} \Psi_I + \frac{\mathrm{i}}{4} \bar{\boldsymbol{D}}^2 \left(z\sigma \frac{\partial \mathbb{K}}{\partial \Phi^I} \right) \ . \tag{6.28b}$$

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A new feature of the 3D foliation of AdS is the appearance of chiral superspace integrals involving an explicit z-derivative. In this dual formulation, the full 4D Lorentz symmetry is no longer manifest. Because the steps we took to construct $\mathbb{K}$ are formally identical to what occurs in four dimensions, we conclude that the target space must be hyperkähler. We will demonstrate this more explicitly in the next section.

6.3 $\mathcal{N} = 2$ supersymmetric σ-models on the cotangent bundles of Hermitian symmetric spaces

The procedure of converting the off-shell σ-model (5.33) to the chiral form (6.26), which we employed in the previous subsection, was purely formal, since we assumed the auxiliary field equations (6.14) to be solved. But the actual solution of this problem is the most difficult part of the construction! In 4D Minkowski space, this problem was solved in a series of papers [18, 19, 20, 21, 22, 23] for a large class of $\mathcal{N} = 2$ supersymmetric σ-models (6.15) in which $K(\Phi, \bar{\Phi})$ is the Kähler potential of a Hermitian symmetric space, and therefore the corresponding curvature tensor is covariantly constant,

$$\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0 \; . \tag{6.29}$$

Here we can immediately apply the results obtained in [18, 19, 20, 21, 22, 23] to the case of σ-models in AdS.

If the Riemann tensor associated with $K(\Phi, \bar{\Phi})$ is covariantly constant, eq. (6.29), then the auxiliary field equations (6.14) are equivalent to the geodesic equation with complex evolution parameter [18, 19]

$$\frac{\mathrm{d}^2 \Upsilon^I(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I{}_{JK}\big(\Upsilon(\zeta), \bar{\Phi}\big) \frac{\mathrm{d}\Upsilon^J(\zeta)}{\mathrm{d}\zeta} \frac{\mathrm{d}\Upsilon^K(\zeta)}{\mathrm{d}\zeta} = 0 \; . \tag{6.30}$$

This equation has a unique solution under the initial conditions

$$\Upsilon^I(0) = \Phi^I \; , \qquad \dot{\Upsilon}^I(0) = \Sigma^I \; . \tag{6.31}$$

In particular, from (6.30) we derive

$$\Upsilon_2^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = -\frac{1}{2}\Gamma^I{}_{JK}\big(\Phi, \bar{\Phi}\big) \Sigma^J \Sigma^K \; , \tag{6.32}$$

with $\Gamma^I{}_{JK}(\Phi, \bar{\Phi})$ the Christoffel symbols for the Kähler metric $g_{I\bar{J}}(\Phi, \bar{\Phi})$. The function Υ_2^I determines the supersymmetry transformation law (6.19).

Upon elimination of the auxiliary superfields, the Lagrangian $\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma})$ appearing in (6.16) can be shown to take the form [23]:

$$\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) = K(\Phi, \bar{\Phi}) - \frac{1}{2}\boldsymbol{\Sigma}^{\mathrm{T}}\boldsymbol{g}\frac{\ln\left(\mathbb{1} + \boldsymbol{R}_{\Sigma,\bar{\Sigma}}\right)}{\boldsymbol{R}_{\Sigma,\bar{\Sigma}}}\boldsymbol{\Sigma}\ , \quad \boldsymbol{\Sigma} := \begin{pmatrix} \Sigma^I \\ \bar{\Sigma}^{\bar{I}} \end{pmatrix}\ , \tag{6.33}$$

where

$$\boldsymbol{R}_{\Sigma,\bar{\Sigma}} := \begin{pmatrix} 0 & (R_\Sigma)^I{}_{\bar{J}} \\ (R_{\bar{\Sigma}})^{\bar{I}}{}_J & 0 \end{pmatrix}\ , \quad (R_\Sigma)^I{}_{\bar{J}} := \frac{1}{2}R_K{}^I{}_{L\bar{J}}\Sigma^K\Sigma^L\ , \quad (R_{\bar{\Sigma}})^{\bar{I}}{}_J := \overline{(R_\Sigma)^I{}_{\bar{J}}}\ , \tag{6.34}$$

and

$$\boldsymbol{g} := \begin{pmatrix} 0 & g_{I\bar{J}} \\ g_{\bar{I}J} & 0 \end{pmatrix}\ . \tag{6.35}$$

A different universal representation for $\mathcal{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma})$ can be found in [22].

The hyperkähler potential (6.27) can be shown [23] to be

$$\mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + \frac{1}{2}\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{g}^{-1}\mathcal{F}\left(-\boldsymbol{R}_{\Psi,\bar{\Psi}}\right)\boldsymbol{\Psi}\ , \tag{6.36}$$

where

$$\mathcal{F}(x) = \frac{1}{x}\left\{\sqrt{1+4x} - 1 - \ln\frac{1 + \sqrt{1+4x}}{2}\right\}\ , \quad \mathcal{F}(0) = 1 \tag{6.37}$$

and the operator $\boldsymbol{R}_{\Psi,\bar{\Psi}}$ is defined as

$$\boldsymbol{R}_{\Psi,\bar{\Psi}} := \begin{pmatrix} 0 & (R_\Psi)_I{}^{\bar{J}} \\ (R_{\bar{\Psi}})_{\bar{I}}{}^J & 0 \end{pmatrix}\ ,$$

$$(R_\Psi)_I{}^{\bar{J}} = (R_\Psi)_{IK}\,g^{K\bar{J}}\ , \quad (R_\Psi)_{KL} := \frac{1}{2}R_K{}^I{}_L{}^J\,\Psi_I\Psi_J\ . \tag{6.38}$$

7 The most general $\mathcal{N} = 2$ supersymmetric σ-model using the 3D foliation

In this section, we attempt to generalize the models discussed in the previous section.

7.1 The most general $\mathcal{N} = 2$ supersymmetric σ-model

We take the action

$$S = \int \frac{dz}{(sz)^2} \left\{ \int d^3x \, d^2\boldsymbol{\theta} \, d^2\bar{\boldsymbol{\theta}} \, \mathbb{K}(\phi, \bar{\phi}) + \left(\int d^3x \, d^2\boldsymbol{\theta} \, i H_a(\phi) \partial_z \phi^a + c.c. \right) \right\} \tag{7.1}$$

where $\mathbb{K}$ is the Kähler potential and H_a is a holomorphic (1,0) form. This action is manifestly 3D $\mathcal{N} = 2$ supersymmetric. We make an ansatz for the transformation law under the full 4D AdS supersymmetry of the form

$$\delta\phi^a = -\xi^{\hat{a}} \partial_{\hat{a}} \phi^a - \xi^\alpha \boldsymbol{D}_\alpha \phi^a + 2\boldsymbol{\sigma} \psi^a - \frac{i}{4} \bar{\boldsymbol{D}}^2 (z\sigma\Omega^a) , \tag{7.2}$$

where $\psi^a = \psi^a(\phi)$ and $\Omega^a = \Omega^a(\phi, \bar{\phi})$ are for the moment arbitrary. Requiring invariance of the action under this transformation implies conditions on both the functions ψ^a and Ω^a. Similarly, closure of the algebra imposes additional restrictions. Ultimately, one discovers that the transformation law must take the form

$$\delta\phi^a = -\xi^{\hat{a}} \partial_{\hat{a}} \phi^a - \xi^\alpha \boldsymbol{D}_\alpha \phi^a - 2\boldsymbol{\sigma} \hat{\omega}^{ab} H_b - \frac{i}{4} \bar{\boldsymbol{D}}^2 (z\sigma \hat{\omega}^{ab} \mathbb{K}_b) , \tag{7.3}$$

where $\hat{\omega}^{ab}$ is an antisymmetric chiral quantity obeying a number of conditions. First, it must be covariantly constant,

$$\nabla_c \hat{\omega}^{ab} = 0 , \qquad \nabla_{\bar{c}} \hat{\omega}^{ab} = \partial_{\bar{c}} \hat{\omega}^{ab} = 0 . \tag{7.4}$$

Secondly, it must obey

$$\hat{\omega}^{ab} \hat{\omega}_{bc} = -\delta^a_c , \qquad \hat{\omega}_{ab} = g_{a\bar{c}} g_{b\bar{d}} \hat{\omega}^{\bar{c}\bar{d}} . \tag{7.5}$$

These two conditions imply that the target space is hyperkähler.

A third requirement is that the holomorphic (2,0) form $\hat{\omega}_{ab}$ must be exact, with its one-form potential given by H_a,

$$\hat{\omega}_{ab} = \partial_a H_b - \partial_b H_a . \tag{7.6}$$

In addition, H_a must be related to a holomorphic U(1) Killing vector

$$V^a := i \hat{\omega}^{ab} H_b , \qquad H_b = i \hat{\omega}_{bc} V^c \tag{7.7}$$

which from its definition can be shown to rotate the complex structure,

$$\mathcal{L}_V \hat{\omega}_{ab} = \nabla_a V^c \hat{\omega}_{cb} + \nabla_b V^c \hat{\omega}_{ac} = i \nabla_a H_b - i \nabla_b H_a = i \hat{\omega}_{ab} , \tag{7.8}$$

as well as the holomorphic one-form H_a,

$$\mathcal{L}_V H_\mathrm{a} = \mathrm{i}\,\mathcal{L}_V \hat{\omega}_\mathrm{ab} V^\mathrm{b} = \mathrm{i}\,H_\mathrm{a} \ . \tag{7.9}$$

Remarkably, when all of these conditions are imposed, the supersymmetry algebra closes *off-shell*. The same behavior was observed in the conventional foliation of AdS [1, 2] discussed in section 3.

The action can be written in terms of the Killing vector V^a,

$$S = \int \frac{\mathrm{d}z}{(sz)^2} \left\{ \int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, \mathrm{d}^2\bar{\boldsymbol{\theta}}\, \mathbb{K}(\phi, \bar{\phi}) + \left(\int \mathrm{d}^3x\, \mathrm{d}^2\boldsymbol{\theta}\, V^\mathrm{a} \hat{\omega}_\mathrm{ab} \partial_z \phi^\mathrm{b} + \mathrm{c.c.} \right) \right\} \ . \tag{7.10}$$

The transformation law for ϕ^a similarly can be rewritten

$$\delta\phi^\mathrm{a} = -\xi^{\hat{a}} \partial_{\hat{a}} \phi^\mathrm{a} - \xi^\alpha \boldsymbol{D}_\alpha \phi^\mathrm{a} + 2\mathrm{i}\boldsymbol{\sigma} V^\mathrm{a} - \frac{\mathrm{i}}{4} \bar{\boldsymbol{D}}^2 (z\sigma \hat{\omega}^\mathrm{ab} \mathbb{K}_\mathrm{b}) \ . \tag{7.11}$$

The action derived in the previous section from projective superspace corresponds to a choice of Darboux coordinates where

$$\phi^\mathrm{a} = (\Phi^I, \Psi_I) \ , \qquad H_\mathrm{a} = (-\Psi_I, 0) \ , \qquad \hat{\omega}^\mathrm{ab} = \hat{\omega}_\mathrm{ab} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \ , \tag{7.12a}$$

$$V = \mathrm{i}\Psi_I \frac{\partial}{\partial \Psi_I} - \mathrm{i}\bar{\Psi}_{\bar{J}} \frac{\partial}{\partial \bar{\Psi}_{\bar{J}}} \ . \tag{7.12b}$$

We know from the earlier works [18, 19, 28] that

$$\mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = K\big(\Phi, \bar{\Phi}\big) + \mathcal{H}\big(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}\big) \ , \tag{7.13}$$

where

$$\mathcal{H}\big(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}\big) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}\big(\Phi, \bar{\Phi}\big) \Psi_{I_1} \ldots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \ldots \bar{\Psi}_{\bar{J}_n} \ ,$$
$$\mathcal{H}^{I\bar{J}}\big(\Phi, \bar{\Phi}\big) = g^{I\bar{J}}\big(\Phi, \bar{\Phi}\big) \ . \tag{7.14}$$

Here the coefficients $\mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}$, for $n > 1$, are tensor functions of the Kähler metric $g_{I\bar{J}}\big(\Phi, \bar{\Phi}\big) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$, the Riemann curvature $R_{I\bar{J}K\bar{L}}\big(\Phi, \bar{\Phi}\big)$ and its covariant derivatives. Using this result, we can compute a Killing potential, $\mathcal{K} = \bar{\mathcal{K}}$, corresponding to the Killing vector field (7.12b). In accordance with [42], it is defined by

$V^a(\phi)\partial_a\mathbb{K}(\phi,\bar\phi) = (\mathrm{i}/2)\mathcal{K}(\phi,\bar\phi) + \lambda(\phi)$, for some holomorphic function λ. In our case, it is immediately seen that $\lambda = 0$ and

$$\mathcal{K}\big(\Phi,\Psi,\bar\Phi,\bar\Psi\big) = 2\sum_{n=1}^{\infty} n\,\mathcal{H}^{I_1\cdots I_n \bar J_1\cdots \bar J_n}\big(\Phi,\bar\Phi\big)\Psi_{I_1}\dots\Psi_{I_n}\bar\Psi_{\bar J_1}\dots\bar\Psi_{\bar J_n}\,. \tag{7.15}$$

It follows that $\mathcal{K}(\phi,\bar\phi)$ is a globally defined function on the hyperkähler target space.

As an example, we can consider $\mathcal{X} = \mathbb{C}P^n$. In standard inhomogeneous coordinates for $\mathbb{C}P^n$, the Kähler potential is

$$K(\Phi,\bar\Phi) = r^2 \ln\left(1 + \frac{1}{r^2}\Phi^L\overline{\Phi^L}\right)\,. \tag{7.16}$$

The hyperkähler potential on $T^*\mathbb{C}P^n$ is known to be (see e.g. [21] for a derivation)

$$\mathbb{K}(\Phi,\Psi,\bar\Phi,\bar\Psi) = K(\Phi,\bar\Phi) + r^2\left\{\sqrt{1+4|\Psi|^2/r^2} - \ln\left(1 + \sqrt{1+4|\Psi|^2/r^2}\right)\right\}\,, \tag{7.17}$$

with $|\Psi|^2 := g^{I\bar J}(\Phi,\bar\Phi)\Psi_I\bar\Psi_{\bar J}$. The Killing potential is

$$\mathcal{K}\big(\Phi,\Psi,\bar\Phi,\bar\Psi\big) = r^2\left(\sqrt{1+4|\Psi|^2/r^2} - 1\right)\,. \tag{7.18}$$

We can now demonstrate that the most general supersymmetric σ-model described above, eq. (7.1), can be derived from a model in projective superspace of the form (5.33). Our starting point is the holomorphic symplectic (2,0) form $\hat\omega = \mathrm{d}H$. According to Darboux's theorem (see, e.g., [58]), locally we can choose new complex coordinates $\phi^{\mathrm a} = (\Phi^I, \Psi_I)$, centred around the origin of $\mathbb{C}^{2n}$, in which the holomorphic (1,0) form H looks like a Liouville form

$$H = -\Psi_I\mathrm{d}\Phi^I\,. \tag{7.19}$$

Then $\hat\omega$ coincides with the canonical symplectic form,

$$\hat\omega = \mathrm{d}\Phi^I \wedge \mathrm{d}\Psi_I\,, \tag{7.20}$$

which is equivalent to (7.12a). As a consequence of eq. (7.7), we also observe that the Killing vector field V takes the form (7.12b). Since the vector field V is Killing, from (7.12b) we conclude that

$$\mathbb{K}(\Phi,\mathrm{e}^{\mathrm{i}\alpha}\Psi,\bar\Phi,\mathrm{e}^{-\mathrm{i}\alpha}\bar\Psi) = \mathbb{K}(\Phi,\Psi,\bar\Phi,\bar\Psi) + \left\{\Lambda(\Phi,\Psi) + \mathrm{c.c.}\right\}\,, \qquad \alpha \in \mathbb{R} \tag{7.21}$$

for some holomorphic function $\Lambda(\Phi,\Psi)$. We assume the Kähler potential $\mathbb{K}(\Phi,\Psi,\bar\Phi,\bar\Psi)$ to be a real analytic function on the coordinate chart chosen. Then, the previous result

tells us that modulo a Kähler transformation we can choose $\mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi})$ to be invariant under the U(1) isometry group generated by V,

$$\mathbb{K}(\Phi, e^{i\alpha}\Psi, \bar{\Phi}, e^{-i\alpha}\bar{\Psi}) = \mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) \ . \tag{7.22}$$

We now consider a submanifold defined by $\Psi_I = \bar{\Psi}_{\bar{J}} = 0$. On this submanifold, we introduce a Kähler potential $K(\Phi, \bar{\Phi})$ as follows:

$$K(\Phi, \bar{\Phi}) := \mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi})\Big|_{\Psi = \bar{\Psi} = 0} \ . \tag{7.23}$$

Associated with $K(\Phi, \bar{\Phi})$ is the desired supersymmetric σ-model (5.33).

7.2 Superpotentials and tri-holomorphic isometries

It is natural to ask whether it is possible to modify the action (7.1) to include a holomorphic superpotential. From experience with the Minkowski limit, we expect the superpotential to be associated with a tri-holomorphic isometry X^{a}, which obeys

$$\mathcal{L}_X \hat{\omega}_{\mathrm{ab}} = 0 \ , \qquad \partial_{\bar{\mathrm{b}}} X^{\mathrm{a}} = 0 \ , \qquad \nabla_{\mathrm{a}} X_{\bar{\mathrm{b}}} + \nabla_{\bar{\mathrm{b}}} X_{\mathrm{a}} = 0 \ . \tag{7.24}$$

The tri-holomorphy requirement implies that X^{a} can be locally written as

$$X^{\mathrm{a}} = -\hat{\omega}^{\mathrm{ab}} \partial_{\mathrm{b}} W \ , \tag{7.25}$$

for a holomorphic function W. (The choice of phase on the right-hand side of this expression is conventional at this point, but we will soon see that this is the correct choice.)

Let us make the additional requirement that the action (7.10) be invariant under the tri-holomorphic isometry $\delta\phi^{\mathrm{a}} = X^{\mathrm{a}}$. This leads to

$$\delta S = \int \frac{\mathrm{d}z}{(sz)^2} \Big\{ \int \mathrm{d}^3 x \, \mathrm{d}^2\boldsymbol{\theta} \, \mathrm{d}^2\bar{\boldsymbol{\theta}} \, \mathcal{L}_X \mathbb{K}$$
$$+ \int \mathrm{d}^3 x \, \mathrm{d}^2\boldsymbol{\theta} \, (\mathcal{L}_X V)^{\mathrm{a}} \hat{\omega}_{\mathrm{ab}} \partial_z \phi^{\mathrm{b}} + \int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\bar{\boldsymbol{\theta}} \, (\mathcal{L}_X V)^{\bar{\mathrm{a}}} \hat{\omega}_{\bar{\mathrm{a}}\bar{\mathrm{b}}} \partial_z \bar{\phi}^{\bar{\mathrm{b}}} \Big\} \ . \tag{7.26}$$

The Kähler term is already invariant up to the real part of a holomorphic superfield, so it vanishes. This leaves the superpotential term, and we find the additional requirement

$$\mathcal{L}_X V = [X, V] = 0 \ . \tag{7.27}$$

Making use of this condition along with (7.8), one can show that there exists a globally defined choice for the holomorphic function W,

$$W = -i \, V^{\mathrm{a}} \hat{\omega}_{\mathrm{ab}} X^{\mathrm{b}} \ , \qquad \mathcal{L}_V W = i \, W \ . \tag{7.28}$$

Let us add to the action (7.10) the term

$$\int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\boldsymbol{\theta} \, \frac{1}{(sz)^3} W + \text{c.c.} \ . \tag{7.29}$$

From our experience with the situation in section 3.2, we expect the addition of the superpotential to correspond to the modification of the U(1) Killing vector, so we postulate the modified transformation law

$$\delta\phi^{\mathrm{a}} = -\xi^{\hat{a}}\partial_{\hat{a}}\phi^{\mathrm{a}} - \xi^{\alpha}\boldsymbol{D}_{\alpha}\phi^{\mathrm{a}} + 2\mathrm{i}\boldsymbol{\sigma}\left(V^{\mathrm{a}} - \frac{1}{2s}X^{\mathrm{a}}\right) - \frac{\mathrm{i}}{4}\bar{\boldsymbol{D}}^2(z\sigma\hat{\omega}^{\mathrm{ab}}\mathbb{K}_{\mathrm{b}}) \ . \tag{7.30}$$

At this point, it is easy to show that the new action and transformation law is equivalent to the old action and transformation law for the choice

$$V'^{\mathrm{a}} = V^{\mathrm{a}} - \frac{1}{2s}X^{\mathrm{a}} \ . \tag{7.31}$$

To prove this, observe that

$$\int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\boldsymbol{\theta} \, \frac{1}{(sz)^2} V'^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\partial_z\phi^{\mathrm{b}}$$

$$= \int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\boldsymbol{\theta} \, \frac{1}{(sz)^2}\left(V^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\partial_z\phi^{\mathrm{b}} - \frac{1}{2s}X^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\partial_z\phi^{\mathrm{b}}\right) \ . \tag{7.32}$$

One can show that $\partial_{\mathrm{a}}W = \hat{\omega}_{\mathrm{ab}}X^{\mathrm{b}}$, and so we have

$$\int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\boldsymbol{\theta} \, \frac{1}{(sz)^2}\left(V^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\partial_z\phi^{\mathrm{b}} + \frac{1}{2s}\partial_{\mathrm{b}}W\partial_z\phi^{\mathrm{b}}\right)$$

$$= \int \mathrm{d}^3 x \, \mathrm{d}z \, \mathrm{d}^2\boldsymbol{\theta} \, \frac{1}{(sz)^2}\left(V^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\partial_z\phi^{\mathrm{b}} + \frac{1}{sz}W\right) \ . \tag{7.33}$$

Note that since X^{a} is tri-holomorphic, we are allowed to add it to V^{a} without modifying any of the conditions that V^{a} obeys.

So the general form of the action with a superpotential

$$S = \int \frac{\mathrm{d}z}{(sz)^2}\left\{\int \mathrm{d}^3 x \, \mathrm{d}^2\boldsymbol{\theta} \, \mathrm{d}^2\bar{\boldsymbol{\theta}} \, \mathbb{K}(\phi, \bar{\phi}) \right.$$

$$\left. + \left[\int \mathrm{d}^3 x \, \mathrm{d}^2\boldsymbol{\theta} \, V^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}\left(\partial_z\phi^{\mathrm{b}} - \frac{\mathrm{i}}{sz}X^{\mathrm{b}}\right) + \text{c.c.}\right]\right\} \ , \tag{7.34}$$

with the AdS isometry

$$\delta\phi^{\mathrm{a}} = -\xi^{\hat{a}}\partial_{\hat{a}}\phi^{\mathrm{a}} - \xi^{\alpha}\boldsymbol{D}_{\alpha}\phi^{\mathrm{a}} + 2\mathrm{i}\boldsymbol{\sigma}\left(V^{\mathrm{a}} - \frac{1}{2s}X^{\mathrm{a}}\right) - \frac{\mathrm{i}}{4}\bar{\boldsymbol{D}}^2(z\sigma\hat{\omega}^{\mathrm{ab}}\mathbb{K}_{\mathrm{b}}) \tag{7.35}$$

is completely equivalent to the original form (7.10) with $V'^{\mathrm{a}} = V^{\mathrm{a}} - X^{\mathrm{a}}/2s$. This equivalence is completely analogous to the situation in the AdS frame, which we will elaborate upon in section 8.

For completeness, we should mention that there is one other possibility for adding a superpotential term; however, its geometric significance is quite different. Let us take the original action (7.10) and add a superpotential term (7.29). However, in contrast to the choice of phase made in (7.25) and (7.28), let us take

$$W = f\, V^{\mathrm{a}} \hat{\omega}_{\mathrm{ab}} X^{\mathrm{b}} \, , \qquad f \in \mathbb{R} \, . \tag{7.36}$$

Note that this still obeys

$$V^{\mathrm{a}} W_{\mathrm{a}} = \mathrm{i}\, W \, , \qquad W_{\mathrm{a}} = \mathrm{i}\, f\, \hat{\omega}_{\mathrm{ab}} X^{\mathrm{b}} \, . \tag{7.37}$$

It turns out that for this choice, the superpotential term and the original action (7.10) are separately invariant under the original isometry (7.11). What then is the physical significance of this choice?

The answer lies in the following observation: there is no barrier to choosing the real proportionality factor f in (7.36) to possess *arbitrary z* dependence! So we arrive at a chiral integral of the form

$$\int \mathrm{d}^3 x\, \mathrm{d}z\, \mathrm{d}^2\boldsymbol{\theta}\, \frac{1}{(sz)^2} V^{\mathrm{a}} \hat{\omega}_{\mathrm{ab}} \left(\partial_z \phi^{\mathrm{b}} + \frac{f(z)}{sz} X^{\mathrm{b}} \right) + \mathrm{c.c.} \tag{7.38}$$

There is an obvious interpretation of this additional term: it is a z-dependent gauge connection which gauges the tri-holomorphic isometry. That is, we may identify

$$\mathcal{D}_z \phi^{\mathrm{b}} := \partial_z \phi^{\mathrm{b}} + \frac{f(z)}{sz} X^{\mathrm{b}} \, . \tag{7.39}$$

Note that this connection is actually pure gauge, and so we can remove it if we perform a gauge transformation

$$\delta_g \phi^{\mathrm{a}} = -\Lambda(z) X^{\mathrm{a}} \, , \qquad \Lambda'(z) = \frac{f(z)}{sz} \, . \tag{7.40}$$

In fact, we should always choose $f(z) = 0$ in this way. Eq. (7.38) for $f(z) \neq 0$ is clearly problematic from the point of view of 4D Lorentz invariance of the component action. Any 4D Lorentz transformation for $f(z) \neq 0$ must be accompanied by a gauge transformation to restore $f(z)$ to this form. While this in a very technical sense respects 4D Lorentz invariance, it clearly violates its spirit. So we will always restrict to the case $f(z) = 0$. We emphasize that this is the natural gauge choice.

The actions considered above emerge automatically when one considers projective superspace Lagrangians with holomorphic isometries on the original Kähler manifold. These become tri-holomorphic isometries on the cotangent bundle. When gauged with the intrinsic vector multiplet prepotential of AdS, the above structure can be shown to emerge, including the bizarre factor $f(z)$. It is to this construction which we now turn.

7.3   Gauged σ-models from projective superspace

We return to the original action

$$S = \frac{1}{2\pi i} \oint_C \frac{d\zeta}{\zeta} \int d^3x \, dz \, d^2\boldsymbol{\theta} \, d^2\bar{\boldsymbol{\theta}} \, \frac{1}{(sz)^2} K(\Upsilon, \check{\Upsilon}) \, , \tag{7.41}$$

written in projective superspace. Let us suppose K possesses a holomorphic isometry X^I under which

$$X^I K_I + \check{X}^{\bar{J}} K_{\bar{J}} = F(\Upsilon) + \check{F}(\check{\Upsilon}) \tag{7.42}$$

where F is a holomorphic function of the arctic superfields Υ^I. Such transformations are symmetries of the action. The transformation $\delta\Upsilon^I = X^I$ on the arctic superfield leads to

$$\delta\Phi^I = X(\Phi)^I \, , \qquad \delta\Psi_I = -\partial_I X(\Phi)^J \Psi_J \tag{7.43}$$

for the cotangent bundle coordinates. Denoting this holomorphic isometry as X^{a} when acting on the complex coordinates $\phi^{\mathrm{a}} = (\Phi^I, \Psi_I)$ of the hyperkähler manifold, it is easy to see that it is tri-holomorphic, $\mathcal{L}_X \hat{\omega}_{\mathrm{ab}} = 0$ where $\hat{\omega}_{\mathrm{ab}}$ is the canonical symplectic form.

Now let us consider gauging the isometry. In section 3, we did this in a manifestly covariant way to avoid dealing specifically with the vector prepotential. This required the addition of a fictitious target space coordinate when the function F discussed above was nonzero. We could follow that same procedure here, but for the sake of clarity we will take an alternative approach where the modifications induced by the gauging are more transparent.

We follow the procedure described in [32], which was based on [45]. From (7.42), it follows that [42]

$$X^I K_I = iD + F \tag{7.44}$$

where D is a real function, the Killing potential. We consider a complexified transformation $\delta\Upsilon^I = -\Lambda X^I$ where Λ is an arctic superfield and introduce a tropical abelian

prepotential V which transforms as $V \to V + i\Lambda - i\check{\Lambda}$. The original Lagrangian K can then be modified to

$$K + V\frac{e^{iVL_{\bar{X}}-1}}{iVL_{\bar{X}}}D = K + VD + \frac{i}{2}V^2 L_{\bar{X}}D + \cdots \tag{7.45}$$

where $L_{\bar{X}}D = \bar{X}^{\bar{J}}\partial_{\bar{J}}D$. This new Lagrangian is the gauged σ-model.

The AdS geometry comes equipped with an intrinsic vector multiplet whose prepotential V_{AdS} is defined up to a gauge transformation. The construction of V_{AdS} is given in appendix B. The relevant details here are that V_{AdS}, when written in terms of the rotated $\boldsymbol{\theta}_i$ and $\bar{\boldsymbol{\theta}}^i$ and projected to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^{\underline{2}} = 0$, is given by

$$V_{\mathrm{AdS}}| = \frac{1}{\zeta}\boldsymbol{\theta}^2\left(\frac{i}{sz} - \Lambda'(z)\right) + \zeta\bar{\boldsymbol{\theta}}^2\left(\frac{i}{sz} + \Lambda'(z)\right) . \tag{7.46}$$

The first term in both sets of parentheses is dictated by the requirement that V_{AdS} possess the correct frozen vector multiplet field strength $W_0 = 1/sz$. The second term turns out to be pure gauge and has no effect on W_0. It is possible to show that the gauge connection associated with V_{AdS}, when projected to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^{\underline{2}} = 0$ obeys $\boldsymbol{A}_\alpha^{\underline{1}}| = \bar{\boldsymbol{A}}_{\alpha\underline{1}}| = A_{\alpha\beta}| = 0$, while $A_z| = \Lambda'(z)$. In other words, from the 3D $\mathcal{N} = 2$ superspace point of view, the connection is pure gauge arising from a z-dependent gauge transformation. We present a proof of this in appendix B for the curious reader.

We can separate $V_{\mathrm{AdS}}|$ into two pieces, $V_{(+)}$ and $V_{(-)}$, which represent the arctic and antarctic components,

$$V_{\mathrm{AdS}}| = V_{(+)} + V_{(-)} , \quad V_{(+)} = \zeta\bar{\boldsymbol{\theta}}^2\left(\frac{i}{sz} + \Lambda'(z)\right) , \quad V_{(-)} = \frac{1}{\zeta}\boldsymbol{\theta}^2\left(\frac{i}{sz} - \Lambda'(z)\right) . \tag{7.47}$$

Note that $V_{(+)}$ and $V_{(-)}$ are nilpotent, $(V_{(+)})^2 = (V_{(-)})^2 = 0$. From this observation it is possible to show that the gauged Lagrangian can be written

$$S = \frac{1}{2\pi i}\oint_C \frac{d\zeta}{\zeta}\int d^3x\,dz\,d^2\boldsymbol{\theta}\,d^2\bar{\boldsymbol{\theta}}\,\frac{1}{(sz)^2}K(\hat{\Upsilon}, \check{\hat{\Upsilon}}) , \qquad \hat{\Upsilon}^I = \Upsilon^I - iV_{(+)}X^I(\Upsilon) . \tag{7.48}$$

The superfield $\hat{\Upsilon}^I$ is the covariant arctic superfield which we discussed in section 3; here we have constructed it explicitly and denoted it with a circumflex. One can easily see its lowest component $\hat{\Phi}^I = \Upsilon_0^I$ remains chiral, but the next component $\hat{\Sigma}^I = \Upsilon_1^I$ obeys a new modified complex linearity constraint,

$$-\frac{1}{4}\bar{\boldsymbol{D}}^2\hat{\Sigma}^I = i\partial_z\hat{\Phi}^I + \left(\frac{1}{sz} + i\Lambda'(z)\right)X^I(\hat{\Phi}) . \tag{7.49}$$

Because $\Lambda'(z)$ may be interpreted as a z-dependent gauge connection A_z, the above expression can be interpreted as

$$-\frac{1}{4}\bar{\boldsymbol{D}}^2\hat{\Sigma}^I = \mathrm{i}\,\mathcal{D}_z\hat{\Phi}^I + \frac{1}{sz}X^I(\hat{\Phi})\,, \qquad \mathcal{D}_z\hat{\Phi}^I = \partial_z\hat{\Phi}^I + \Lambda'(z)X^I(\hat{\Phi})\,. \tag{7.50}$$

The elimination of auxiliaries proceeds exactly as before and the form of the Lagrangian on the tangent bundle is unchanged, except for the modification to the complex linearity constraint. However, upon dualizing to the cotangent bundle, we find

$$\begin{aligned}
S_{\mathrm{dual}} &= \int \mathrm{d}^3x\,\mathrm{d}z\,\mathrm{d}^2\boldsymbol{\theta}\,\mathrm{d}^2\bar{\boldsymbol{\theta}}\,\frac{1}{(sz)^2}\mathbb{K}(\hat{\Phi},\hat{\Psi},\hat{\bar{\Phi}},\hat{\bar{\Psi}}) \\
&\quad - \left[\int \mathrm{d}^3x\,\mathrm{d}z\,\mathrm{d}^2\boldsymbol{\theta}\,\frac{1}{(sz)^2}\hat{\Psi}_I\left(\mathrm{i}\,\mathcal{D}_z\hat{\Phi}^I + \frac{1}{sz}X^I(\hat{\Phi})\right) + \mathrm{c.c.}\right]
\end{aligned} \tag{7.51}$$

where as before

$$\mathbb{K}(\hat{\Phi},\hat{\Psi},\hat{\bar{\Phi}},\hat{\bar{\Psi}}) := \mathcal{L}(\hat{\Phi},\hat{\Sigma},\hat{\bar{\Phi}},\hat{\bar{\Sigma}}) + \hat{\Sigma}^I\hat{\Psi}_I + \hat{\bar{\Sigma}}^{\bar{J}}\hat{\bar{\Psi}}_{\bar{J}} \tag{7.52}$$

When recast into general chiral coordinates, this Lagrangian matches that postulated in (7.34).

To complete the equivalence, we must determine the modified transformation laws for $\hat{\Phi}^I$ and $\hat{\Psi}_I$. Letting $\hat{e}$ correspond to the U(1) generator, we define a covariant weight-zero arctic multiplet $\hat{\Upsilon}$ by

$$\hat{\Upsilon} = \exp\left(V_{(+)}\hat{e}\right)\Upsilon\,. \tag{7.53}$$

(For the arctic multiplets we have been discussing, $\mathrm{i}\hat{e}\Upsilon^I = X^I$.) The corresponding covariant derivatives are $\boldsymbol{\mathcal{D}}_A$. The covariant arctic multiplet $\hat{\Upsilon}$ is defined to transform covariantly,

$$\delta\hat{\Upsilon} = -\left(\xi^A\boldsymbol{\mathcal{D}}_A + \lambda^{ij}J_{ij} + \mathrm{i}\,\Gamma\hat{e}\right)\hat{\Upsilon}\,, \tag{7.54}$$

where Γ is real, while the original arctic multiplet transforms as

$$\delta\Upsilon = -\left(\xi^A\boldsymbol{D}_A + \lambda^{ij}J_{ij} + \mathrm{i}\,\Lambda\hat{e}\right)\hat{\Upsilon}\,, \tag{7.55}$$

where Λ is arctic. Together, these imply that $V_{(+)}$ should transform as

$$\delta V_{(+)} = -(\xi^A\boldsymbol{D}_A + \lambda^{ij}J_{ij})V_{(+)} - \mathrm{i}\,(\xi^B\boldsymbol{A}_B + \Gamma - \Lambda)\,. \tag{7.56}$$

Note that this implies that V transforms as a projective multiplet of weight zero, up to a Λ-gauge transformation

$$\delta V = -(\xi^A \boldsymbol{D}_A + \lambda^{ij} J_{ij})V + \mathrm{i}\,(\Lambda - \check{\Lambda})\;. \tag{7.57}$$

From the discussion of the intrinsic vector multiplet in section 2, we know that Γ is given in the AdS frame by $\Gamma = 2\varepsilon$. Repeating that analysis in the Minkowski frame, that relation becomes[19]

$$\Gamma| = -\frac{1}{s}\Lambda\;. \tag{7.58}$$

This leads to

$$\delta\hat{\Phi}^I = -\xi^{\hat{a}}\partial_{\hat{a}}\hat{\Phi}^I - \xi^\alpha \boldsymbol{D}_\alpha\hat{\Phi}^I + \bar{\rho}_\alpha\bar{\boldsymbol{D}}^\alpha\hat{\Sigma}^I - \xi^z\mathcal{D}_z\hat{\Phi}^I + \frac{\Lambda}{s}\mathrm{i}\hat{e}\hat{\Phi}^I \tag{7.59a}$$

$$\delta\hat{\Sigma}^I = -\xi^{\hat{a}}\partial_{\hat{a}}\hat{\Sigma}^I - \xi^\alpha \boldsymbol{D}_\alpha\hat{\Sigma}^I - \bar{\xi}_\alpha\bar{\boldsymbol{D}}^\alpha\hat{\Sigma}^I - \rho^\alpha \boldsymbol{D}_\alpha\hat{\Phi}^I + \bar{\rho}_\alpha\bar{\boldsymbol{D}}^\alpha\hat{\Upsilon}_2^I$$
$$- \xi^z\mathcal{D}_z\hat{\Sigma}^I + 2\mathrm{i}\Lambda\hat{\Sigma}^I + \frac{\Lambda}{s}\mathrm{i}\hat{e}\hat{\Sigma}^I \tag{7.59b}$$

where

$$\mathrm{i}\hat{e}\hat{\Phi}^I = X^I(\hat{\Phi})\;, \qquad \mathrm{i}\hat{e}\hat{\Sigma}^I = \hat{\Sigma}^J\partial_J X^I(\hat{\Phi})\;. \tag{7.60}$$

The additional terms we have added to $\delta\hat{\Phi}^I$ lead to

$$\delta\hat{\Phi}^I = -\xi^A D_A\hat{\Phi}^I + \frac{\boldsymbol{\sigma}}{s}\hat{e}\hat{\Phi}^I - \frac{\mathrm{i}z}{4}\bar{\boldsymbol{D}}^2(\sigma\hat{\Sigma}^I) \tag{7.61}$$

which is manifestly chiral. Note the appearance of the new gauged isometry term. Performing the duality in the usual way leads to a similar modification for $\delta\hat{\Psi}_I$, and we find

$$\delta\hat{\Phi}^I = -\xi^A\mathcal{D}_A\hat{\Phi}^I + \frac{\boldsymbol{\sigma}}{s}\hat{e}\hat{\Phi}^I - \frac{\mathrm{i}z}{4}\bar{\boldsymbol{D}}^2\left(\sigma\frac{\partial\mathbb{K}}{\partial\hat{\Psi}_I}\right)\;, \tag{7.62a}$$

$$\delta\hat{\Psi}_I = -\xi^A\mathcal{D}_A\hat{\Psi}_I - 2\boldsymbol{\sigma}\hat{\Psi}_I + \frac{\boldsymbol{\sigma}}{s}\hat{e}\hat{\Psi}_I + \frac{\mathrm{i}z}{4}\bar{\boldsymbol{D}}^2\left(\sigma\frac{\partial\mathbb{K}}{\partial\hat{\Phi}^I}\right)\;, \tag{7.62b}$$

where $\mathrm{i}\hat{e}\hat{\Psi}_I = -\partial_I X^J(\hat{\Phi})\,\hat{\Psi}_J$. These are the isometries of the action (7.51). Generalizing the action and the transformation law to general chiral coordinates, we find (7.34) and (7.35).

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[19]Calculating Γ is a bit more technical in the Minkowski frame than in the AdS frame. In particular, there are additional more complicated terms in (7.58), which vanish when we project to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^2 = 0$. But (7.58) is the only result we require.

8 Hyperkähler geometry for σ-models in AdS

In the preceding sections, we have mainly been interested in 4D σ-models with $\mathcal{N} = 2$ supersymmetry drawing inspiration from those which emerge naturally from projective superspace descriptions of AdS$^{4|8}$. In rewriting these models in terms of a superspace with only four Grassmann coordinates, there are two possible choices: the conventional AdS frame corresponding to (1.3a), and the 3D foliated frame corresponding to (1.3b). However, no matter which intermediate superspace we choose, it is clear that the component actions must be identical after eliminating the remaining auxiliaries and the target spaces must also be identical. For that reason, we focus in this section on demonstrating the *general* features of a hyperkähler target space $\mathcal{M}$ with a U(1) Killing vector V^μ which rotates the complex structures. We collect a number of geometric results and give the equivalent objects in both the AdS and 3D foliated pictures. Background information on hyperkähler geometry can be found, e.g., in [4, 45].

8.1 General structure of the σ-model target spaces

Let $\mathcal{M}$ be a hyperkähler manifold equipped with three complex structures $(\mathcal{J}_A)^\mu{}_\nu$ along with a U(1) isometry V^μ which acts as a rotation on them. Without loss of generality, we take

$$\mathcal{L}_V \mathcal{J}_1 = -\mathcal{J}_2 , \quad \mathcal{L}_V \mathcal{J}_2 = +\mathcal{J}_1 , \quad \mathcal{L}_V \mathcal{J}_3 = 0 . \tag{8.1}$$

The three Kähler two-forms are

$$\Omega_A = \frac{1}{2}(\Omega_A)_{\mu\nu} \, \mathrm{d}\phi^\mu \wedge \mathrm{d}\phi^\nu , \qquad (\Omega_A)_{\mu\nu} = g_{\mu\rho}(\mathcal{J}_A)^\rho{}_\nu . \tag{8.2}$$

From Ω_1 and Ω_2 we construct the complex $(2,0)$ and $(0,2)$ forms with respect to $\mathcal{J}_3$

$$\Omega_\pm = \frac{1}{2}\Omega_1 \pm \frac{\mathrm{i}}{2}\Omega_2 , \qquad \mathcal{L}_V \Omega_\pm = \pm \mathrm{i}\, \Omega_\pm. \tag{8.3}$$

Ω_+ is holomorphic with respect to $\mathcal{J}_3$.

Each of these two-forms is closed by construction. Due to the properties of the Killing vector V^μ, it turns out that complex structures Ω_+ and Ω_- (and any linear combination) are actually exact. This is easily proven. Consider $\rho_+ := -\mathrm{i}\, \imath_V \Omega_+$, which is a holomorphic $(1,0)$ form with respect to $\mathcal{J}_3$. It is a simple exercise to show that $\mathrm{d}\rho_+ = \Omega_+$. Similarly, $\rho_- := +\mathrm{i}\, \imath_V \Omega_-$ obeys $\mathrm{d}\rho_- = \Omega_-$. It follows that $\rho_1 = \rho_+ + \rho_-$ and $\rho_2 = -\mathrm{i}(\rho_+ - \rho_-)$

are the potentials for Ω_1 and Ω_2, and they are given by $\rho_1 = \imath_V \Omega_2$ and $\rho_2 = -\imath_V \Omega_1$. Note however that the third Kähler two-form Ω_3 need not be exact. Because some of the Kähler two-forms are exact, $\mathcal{M}$ must be a non-compact manifold.

Because V^μ is holomorphic with respect to $\mathcal{J}_3$, we may introduce a real Killing potential $\mathcal{K}$ [42, 45]

$$V^\mu = \frac{1}{2}(\mathcal{J}_3)^\mu{}_\nu \nabla^\nu \mathcal{K} \tag{8.4}$$

which is defined up to a constant shift. It is straightforward to show that $\mathcal{K}$ is the Kähler potential with respect to $\mathcal{J}_1$ and $\mathcal{J}_2$ and indeed any complex structure $\mathcal{J}_\perp$ which is perpendicular to $\mathcal{J}_3$. In other words,

$$g_{\mu\nu} = \frac{1}{2}\nabla_\mu \nabla_\nu \mathcal{K} + \frac{1}{2}(\mathcal{J}_\perp)_\mu{}^\rho (\mathcal{J}_\perp)_\nu{}^\sigma \nabla_\rho \nabla_\sigma \mathcal{K} \ . \tag{8.5}$$

Because $\nabla_\mu \mathcal{K}$ is a globally defined one-form, this implies that the Ω_1 and Ω_2 are exact, $\Omega_1 = \mathrm{d}\rho_1$ and $\Omega_2 = \mathrm{d}\rho_2$ with

$$\rho_1 = \frac{1}{2}\nabla_\mu \mathcal{K} \,(\mathcal{J}_1)^\mu{}_\nu \mathrm{d}\phi^\nu \ , \qquad \rho_2 = \frac{1}{2}\nabla_\mu \mathcal{K} \,(\mathcal{J}_2)^\mu{}_\nu \mathrm{d}\phi^\nu \ . \tag{8.6}$$

These may be alternatively written

$$\rho_1 = V_\mu \,(\mathcal{J}_2)^\mu{}_\nu \mathrm{d}\phi^\nu \ , \qquad \rho_2 = -V_\mu \,(\mathcal{J}_1)^\mu{}_\nu \mathrm{d}\phi^\nu \ . \tag{8.7}$$

Let us now suppose the space is equipped with a tri-holomorphic isometry X^μ. It follows that X^μ is associated with three distinct Killing potentials $D_{(A)}$,

$$X^\mu = \frac{1}{2}(\mathcal{J}_A)^\mu{}_\nu \nabla^\nu D_{(A)} \qquad \text{(no summation on A)} \ . \tag{8.8}$$

One can show that if $\mathcal{J}_{A\perp}$ is some complex structure orthogonal to $\mathcal{J}_A$, then $D_{(A)}$ is the real part of a holomorphic function with respect to $\mathcal{J}_{A\perp}$,

$$0 = \nabla_\mu \nabla_\nu D_{(A)} + (\mathcal{J}_{A\perp})_\mu{}^\rho (\mathcal{J}_{A\perp})_\nu{}^\sigma \nabla_\rho \nabla_\sigma D_{(A)} \ . \tag{8.9}$$

Some of these Killing potentials possess elegant geometric definitions if we specialize to the case where $[V, X] = 0$. Consider the set of real functions

$$F_{(A)} = V^\mu(\Omega_A)_{\mu\nu} X^\nu = \imath_X \imath_V \Omega_A \ . \tag{8.10}$$

One can show, using Cartan's formula $\mathcal{L}_X = \imath_X \mathrm{d} + \mathrm{d}\, \imath_X$ and the identity $\mathcal{L}_X \imath_Y - \imath_Y \mathcal{L}_X = \imath_{[X,Y]}$, that

$$\mathrm{d}F_{(A)} = -\imath_X \mathcal{L}_V \Omega_A \quad \Longleftrightarrow \quad \nabla_\mu F_{(A)} = -X^\nu \mathcal{L}_V (\Omega_A)_{\nu\mu} \tag{8.11}$$

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or equivalently

$$\mathrm{d}F_{(1)} = \frac{1}{2}\mathrm{d}D_{(2)} \,, \qquad \mathrm{d}F_{(2)} = -\frac{1}{2}\mathrm{d}D_{(1)} \,, \qquad \mathrm{d}F_{(3)} = 0 \,. \tag{8.12}$$

The first two relations tell us that $D_{(1)}$ and $D_{(2)}$ can be chosen as

$$D_{(1)} := -2V^{\mu}(\Omega_2)_{\mu\nu}X^{\nu} \,, \qquad D_{(2)} := 2V^{\mu}(\Omega_1)_{\mu\nu}X^{\nu} \,. \tag{8.13}$$

Remarkably, the Killing potentials for X for any complex structure orthogonal to $\mathcal{J}_3$ are purely geometric quantities (i.e. globally defined scalar fields). The third relation in (8.12) means that $F_{(3)}$ must be constant and it is easy to see that it must be given by

$$F_{(3)} = \frac{1}{2}\mathcal{L}_X\mathcal{K} = \mathrm{const} \,. \tag{8.14}$$

We have seen that in models derived from projective superspace, the constant in (8.14) is actually zero. It will be shown in subsection 8.2 that this constant is always zero.

We may construct a new $\mathrm{U}(1)$ vector $V'^{\mu} = V^{\mu} + rX^{\mu}$ for $r \in \mathbb{R}$, which also rotates the complex structures. The constant r is determined by our normalization convention for X^{μ}. The choices we have made in earlier sections amount to $r = -1/2s$. It is easy to see that the corresponding Killing potential is $\mathcal{K}' = \mathcal{K} + rD_{(3)}$. This is just a Kähler-like transformation: $\mathcal{K}$ is the Kähler potential and $D_{(3)}$ is the real part of a holomorphic function with respect to any complex structure orthogonal to $\mathcal{J}_3$. Similarly, the shift in V'^{μ} induces gauge transformations in the one-forms

$$\rho'_1 = \rho_1 + \frac{r}{2}\mathrm{d}D_{(2)} \,, \qquad \rho'_2 = \rho_2 - \frac{r}{2}\mathrm{d}D_{(1)} \,. \tag{8.15}$$

Of course, Ω_1 and Ω_2 are unchanged by this shift.

8.2 Geometry in the 3D foliated frame

Let us now specialize to 3D foliated frame, where the complex structures are

$$\mathbb{J}_1 = \begin{pmatrix} 0 & \hat{\omega}^{\mathrm{a}}{}_{\bar{\mathrm{b}}} \\ \hat{\omega}^{\bar{\mathrm{a}}}{}_{\mathrm{b}} & 0 \end{pmatrix} \,, \quad \mathbb{J}_2 = \begin{pmatrix} 0 & \mathrm{i}\,\hat{\omega}^{\mathrm{a}}{}_{\bar{\mathrm{b}}} \\ -\mathrm{i}\,\hat{\omega}^{\bar{\mathrm{a}}}{}_{\mathrm{b}} & 0 \end{pmatrix} \,, \quad \mathbb{J}_3 = \begin{pmatrix} \mathrm{i}\,\delta^{\mathrm{a}}{}_{\mathrm{b}} & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar{\mathrm{a}}}{}_{\bar{\mathrm{b}}} \end{pmatrix} \,, \tag{8.16}$$

where V^{μ} acts on them as in (8.1). We may choose $\mathcal{J}_3 = \mathbb{J}_3$. However, it is not completely obvious that $\mathbb{J}_1$ and $\mathbb{J}_2$ should be identified with $\mathcal{J}_1$ and $\mathcal{J}_2$. The most we can say is that they are identified up to some rotation. Let us assume $\mathcal{J}_1 = \mathbb{J}_1$ and $\mathcal{J}_2 = \mathbb{J}_2$.

Here the Kähler two-forms have the conventional form

$$\Omega_1 = \frac{1}{2}\hat{\omega}_{\mathrm{ab}}\,\mathrm{d}\phi^{\mathrm{a}}\wedge\mathrm{d}\phi^{\mathrm{b}} + \frac{1}{2}\hat{\omega}_{\bar{\mathrm{a}}\bar{\mathrm{b}}}\,\mathrm{d}\bar{\phi}^{\bar{\mathrm{a}}}\wedge\mathrm{d}\bar{\phi}^{\bar{\mathrm{b}}}\ , \tag{8.17a}$$

$$\Omega_2 = -\frac{\mathrm{i}}{2}\hat{\omega}_{\mathrm{ab}}\,\mathrm{d}\phi^{\mathrm{a}}\wedge\mathrm{d}\phi^{\mathrm{b}} + \frac{\mathrm{i}}{2}\hat{\omega}_{\bar{\mathrm{a}}\bar{\mathrm{b}}}\,\mathrm{d}\bar{\phi}^{\bar{\mathrm{a}}}\wedge\mathrm{d}\bar{\phi}^{\bar{\mathrm{b}}}\ , \tag{8.17b}$$

$$\Omega_3 = -\mathrm{i}g_{\mathrm{a}\bar{\mathrm{b}}}\,\mathrm{d}\phi^{\mathrm{a}}\wedge\mathrm{d}\bar{\phi}^{\bar{\mathrm{b}}}\ , \tag{8.17c}$$

with

$$\Omega_+ = \frac{1}{2}\hat{\omega}_{\mathrm{ab}}\,\mathrm{d}\phi^{\mathrm{a}}\wedge\mathrm{d}\phi^{\mathrm{b}}\ , \qquad \Omega_- = \frac{1}{2}\hat{\omega}_{\bar{\mathrm{a}}\bar{\mathrm{b}}}\,\mathrm{d}\bar{\phi}^{\bar{\mathrm{a}}}\wedge\mathrm{d}\bar{\phi}^{\bar{\mathrm{b}}}\ . \tag{8.18}$$

Ω_+ and Ω_- are exact (and similarly Ω_1 and Ω_2) with

$$\rho_+ = H_{\mathrm{a}}\mathrm{d}\phi^{\mathrm{a}}\ , \qquad \rho_- = \bar{H}_{\bar{\mathrm{a}}}\mathrm{d}\bar{\phi}^{\bar{\mathrm{a}}}\ . \tag{8.19}$$

These are holomorphic and anti-holomorphic with respect to $\mathbb{J}_3$. We recall that the holomorphic Killing vector field $V^{\mathrm{a}}(\phi)$ is related to $H_{\mathrm{a}}(\phi)$ according to eq. (7.7).

Let us now demonstrate that the constant in (8.14) is actually zero. As argued at the end of subsection 7.1, we can always choose local complex coordinates $\phi^{\mathrm{a}} = (\Phi^I, \Psi_I)$ on $\mathcal{M}$ such that H_{a} and V^{a} have the form (7.12a) and (7.12b) respectively. In these Darboux coordinates, the hyperkähler potential $\mathbb{K}(\phi, \bar{\phi})$ is given by eqs. (7.13) and (7.14). Moreover, for the Killing potential $\mathcal{K}(\phi, \bar{\phi})$ associated with V we derived the explicit expression (7.15). Since $\mathcal{K}$ is at least quadratic in ϕ's and $\bar{\phi}$'s, the function $\mathcal{L}_X\mathcal{K}$ is at least linear in the superfield variables (i.e. no constant term is present in the Taylor series for $\mathcal{L}_X\mathcal{K}$). As a result, the only option for the relation (8.14) is

$$\mathcal{L}_X\mathcal{K} = 0\ . \tag{8.20}$$

If we have a tri-holomorphic Killing vector X^{a}, we may conventionally choose

$$X^{\mathrm{a}} = -\hat{\omega}^{\mathrm{ab}}W_{\mathrm{b}} \tag{8.21}$$

for some holomorphic function W. It follows that

$$D_{(1)} = -2(W + \bar{W})\ , \qquad D_{(2)} = 2\mathrm{i}(W - \bar{W})\ , \tag{8.22}$$

where

$$W = -\mathrm{i}\,V^{\mathrm{a}}\hat{\omega}_{\mathrm{ab}}X^{\mathrm{b}}\ . \tag{8.23}$$

8.3 Geometry in the AdS frame

The complex structures for this geometry were given in (1.10) and (1.11). However, we are free to choose a different basis for the complex structures. To make contact with the assumptions made in eq. (8.1), let us choose $\mathcal{J}_3$ to be the invariant complex structure,

$$\mathcal{J}_3 = J_1 \cos\theta + J_2 \sin\theta = \frac{1}{|\mu|} \begin{pmatrix} 0 & \mu\omega^a{}_{\bar{b}} \\ \bar{\mu}\omega^{\bar{a}}{}_b & 0 \end{pmatrix} . \tag{8.24}$$

We are free to choose $\mathcal{J}_1$ and $\mathcal{J}_2$ however we like, provided we maintain the conditions (8.1). The simplest choice is to let one of them be the diagonal complex structure, say

$$\mathcal{J}_1 = J_3 = \begin{pmatrix} \mathrm{i}\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\delta^{\bar{a}}{}_{\bar{b}} \end{pmatrix} . \tag{8.25}$$

The other is then found to be

$$\mathcal{J}_2 = J_1 \sin\theta - J_2 \cos\theta = \frac{1}{|\mu|} \begin{pmatrix} 0 & -\mathrm{i}\mu\omega^a{}_{\bar{b}} \\ \mathrm{i}\bar{\mu}\omega^{\bar{a}}{}_b & 0 \end{pmatrix} . \tag{8.26}$$

It is a simple exercise to check that this choice for the complex structures respects the quaternionic algebra.

We easily see that V^μ given by (8.4) is indeed

$$V^a = \frac{\mu}{2|\mu|}\omega^{ab}\mathcal{K}_b , \qquad V^{\bar{a}} = \frac{\bar{\mu}}{2|\mu|}\omega^{\bar{a}\bar{b}}\mathcal{K}_{\bar{b}} . \tag{8.27}$$

By construction, $\mathcal{K}$ is the Kähler potential with respect to $\mathcal{J}_1$; moreover, it must also be the Kähler potential with respect to any complex structure orthogonal to $\mathcal{J}_3$. As discussed in the previous subsection, $\mathcal{K}$ is also a globally defined function.

The first Kähler two-form is the usual Kähler form,

$$\Omega_1 = -\mathrm{i}g_{a\bar{b}}\mathrm{d}\varphi^a \wedge \mathrm{d}\bar{\varphi}^{\bar{b}} . \tag{8.28}$$

It is exact, $\Omega_1 = \mathrm{d}\rho_1$, with

$$\rho_1 = \frac{\mathrm{i}}{2}\mathcal{K}_a\mathrm{d}\varphi^a - \frac{\mathrm{i}}{2}\mathcal{K}_{\bar{b}}\mathrm{d}\bar{\varphi}^{\bar{b}} . \tag{8.29}$$

The second is

$$\Omega_2 = \frac{\mathrm{i}\bar{\mu}}{|\mu|}\omega_{ab}\,\mathrm{d}\varphi^a \wedge \mathrm{d}\varphi^b - \frac{\mathrm{i}\mu}{|\mu|}\omega_{\bar{a}\bar{b}}\,\mathrm{d}\bar{\varphi}^{\bar{a}} \wedge \mathrm{d}\bar{\varphi}^{\bar{b}} . \tag{8.30}$$

Remarkably, this is also exact, with

$$\rho_2 = \frac{i\bar{\mu}}{2|\mu|}\mathcal{K}_{\bar{a}}\omega^{\bar{a}}{}_b d\varphi^b - \frac{i\mu}{2|\mu|}\mathcal{K}_a \omega^a{}_{\bar{b}} d\bar{\varphi}^{\bar{b}} = -iV_a d\varphi^a + iV_{\bar{b}}d\bar{\varphi}^{\bar{b}} \ . \tag{8.31}$$

However, the third Kähler two-form,

$$\Omega_3 = \frac{\bar{\mu}}{|\mu|}\omega_{ab}\, d\varphi^a \wedge d\varphi^b + \frac{\mu}{|\mu|}\omega_{\bar{a}\bar{b}}\, d\bar{\varphi}^{\bar{a}} \wedge d\bar{\varphi}^{\bar{b}} \ , \tag{8.32}$$

is not exact.

Suppose again that X^a is a holomorphic isometry which commutes with V^a. We know that there must be two Killing potentials with purely geometric definitions. The first is

$$D_{(1)} = -iX^a \mathcal{K}_a + iX^{\bar{a}}\mathcal{K}_{\bar{a}} \ . \tag{8.33}$$

This is the usual Killing potential for a holomorphic isometry since $\mathcal{J}_1$ is the diagonal complex structure. The remaining two Killing potentials, $D_{(2)}$ and $D_{(3)}$ must be the real part of a holomorphic field. Because X^a is tri-holomorphic, we may introduce a holomorphic function W, with

$$X^a = -\omega^{ab}W_b \ . \tag{8.34}$$

It is straightforward to show that (up to a constant)

$$D_{(2)} = -2i|\mu|\left(\frac{W}{\mu} - \frac{\bar{W}}{\bar{\mu}}\right) \ , \qquad D_{(3)} = -2|\mu|\left(\frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}}\right) \ . \tag{8.35}$$

In particular, the shift $V \to V - X/2|\mu|$ does indeed correspond to a redefinition of the AdS Lagrangian by $\mathcal{K} \to \mathcal{K} + W/\mu + \bar{W}/\bar{\mu}$.

The second Killing potential with a purely geometric definition is

$$D_{(2)} = 2iX^a V_a - 2iX^{\bar{a}}V_{\bar{a}} \ . \tag{8.36}$$

It follows that

$$-2i|\mu|\left(\frac{W}{\mu} - \frac{\bar{W}}{\bar{\mu}}\right) = 2iX^a V_a - 2iX^{\bar{a}}V_{\bar{a}} \ , \tag{8.37}$$

which we can interpret as a definition of the imaginary part of W/μ. There is no corresponding definition for the real part.

9 $\mathcal{N} = 2$ AdS supersymmetric σ-model on $T^*\mathbb{C}P^n$

For the general off-shell σ-model (1.26), the two schemes to eliminate the auxiliary superfields, which we presented in sections 3 and 6, were purely formal. Indeed, in sections 3 and 6 we assumed that the problems of solving the auxiliary field equations (3.12) and (6.14), which originate in the AdS frame and in the 3D foliated frame respectively, had been solved. In practice, however, these equations are very difficult to solve. In the 3D foliated frame, solutions to the equations (6.14) are known for a large class of $\mathcal{N} = 2$ supersymmetric σ-models in AdS_4 in which the fields take their values in the cotangent bundles of Hermitian symmetric spaces. These solutions were briefly described in subsection 6.3. As concerns the AdS frame, no exact solution of the auxiliary field equations (3.12) is known except for the case when $K(\Phi, \bar{\Phi})$ corresponds to a flat Kähler space. Here we consider a specific example, the σ-model on the cotangent bundle of $\mathbb{C}P^n$, where the problem of solving the auxiliary field equations (3.12) can be bypassed, and the hyperkähler potential, $\mathcal{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi})$, in (3.18) can be found by using superconformal techniques developed in [59].

We recall that the manifestly $\mathcal{N} = 2$ supersymmetric action in AdS_4, eq. (1.23), can be reduced to $\mathcal{N} = 1$ AdS superspace [10]. The result is eq. (3.1), which we repeat here for convenience,

$$S = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \, \mathcal{L} \; , \qquad \mathcal{L} = \oint_C \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \mathcal{L}^{[2]}(\zeta) \Big| \; , \tag{9.1}$$

where we have introduced the Lagrangian $\mathcal{L}^{[2]}(\zeta)$ defined[20] by

$$\mathcal{L}^{(2)}(v) = \mathrm{i}(v^{\underline{1}})^2 \zeta \mathcal{L}^{[2]}(\zeta) \tag{9.2}$$

and made use of the bar-projection (1.7).

Let us consider a special $\mathcal{N} = 2$ supersymmetric field theory in AdS_4 describing a tensor multiplet $\mathcal{H}^{(2)}$,

$$\mathcal{H}^{(2)}(v) = \breve{\mathcal{H}}^{(2)}(v) = \mathcal{H}^{ij} v_i v_j \; , \qquad \mathcal{D}_\alpha^{(k} \mathcal{H}^{ij)} = \bar{\mathcal{D}}_{\dot\alpha}^{(k} \mathcal{H}^{ij)} = 0 \; , \tag{9.3}$$

coupled to a system of weight-zero arctic multiplets Υ^I and their smile-conjugates $\breve{\Upsilon}^{\bar{I}}$. The Lagrangian is

$$\mathcal{L}^{(2)} = \mathcal{H}^{(2)} K(\Upsilon, \breve{\Upsilon}) \; , \tag{9.4}$$

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[20]The inhomogeneous complex coordinate for the north chart of $\mathbb{C}P^1$, ζ, is defined as usual: $v^i = v^{\underline{1}}(1, \zeta)$.

where $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ is the Kähler potential of a real analytic Kähler manifold $\mathcal{X}$. If we freeze the tensor multiplet,

$$\mathcal{H}^{(2)} \longrightarrow \frac{1}{2s}\mathcal{S}^{(2)} \tag{9.5}$$

then the Lagrangian reduces to that describing the σ-model (1.26). Note that the action is invariant under the Kähler transformations (1.27).

Upon projection to $\mathcal{N} = 1$ AdS superspace, $\mathcal{H}^{[2]}(\zeta)$ reads[21]

$$\mathcal{H}^{[2]}(\zeta) = \frac{1}{\zeta}\chi + G - \zeta\bar{\chi} , \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\chi = 0 , \qquad (\bar{\mathcal{D}}^2 - 4\mu)G = 0 , \quad \bar{G} = G . \tag{9.6}$$

For the arctic multiplets we get

$$\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I = \Phi^I + \zeta\, \Sigma^I + O(\zeta^2) , \qquad \bar{\mathcal{D}}_{\dot{\alpha}}\Phi^I = 0 , \quad (\bar{\mathcal{D}}^2 - 4\mu)\Sigma^I = 0 . \tag{9.7}$$

The $\mathcal{N} = 1$ AdS superfields Υ_2^I, $\Upsilon_3^I, \ldots$, are complex unconstrained.

The theory (9.4) is $\mathcal{N} = 2$ superconformal. Since AdS$^{4|8}$ is conformally flat, the theory can be re-formulated in $\mathcal{N} = 2$ Minkowski superspace where its Lagrangian has essentially the same form, $L^{(2)} = H^{(2)}K(\Upsilon, \breve{\Upsilon})$, but the supermultiplets $H^{(2)}$, Υ and $\breve{\Upsilon}$ are projective with respect to the flat covariant derivatives. In Minkowski superspace, the auxiliary superfields have been eliminated in [59] for one particular case of $\mathcal{X}$ – the complex projective space $\mathbb{C}P^n$. Here we can use the flat-superspace results of [59] and then lift them to AdS using the consideration of superconformal invariance. This will allow us to obtain a formulation in terms of $\mathcal{N} = 1$ chiral superfields for the σ-model (1.26) in the case $\mathcal{X} = \mathbb{C}P^n$. In other words, we can use the same superconformal model to derive the hyperkähler potential for $T^*\mathbb{C}P^n$ both in the AdS frame and in the 3D foliated frame by making a different choice of $\mathcal{S}^{[2]}$,

$$\mathcal{H}^{[2]} = \frac{1}{2s}\mathcal{S}^{[2]} = \begin{cases} \dfrac{i\mu}{2|\mu|}\dfrac{1}{\zeta} + \dfrac{i\bar{\mu}}{2|\mu|}\zeta , & \text{AdS frame ;} \\ 1 , & \text{3D foliated frame .} \end{cases} \tag{9.8}$$

Using standard inhomogeneous coordinates for $\mathbb{C}P^n$, the Kähler potential and the metric are

$$K(\Phi, \bar{\Phi}) = r^2 \ln\left(1 + \frac{1}{r^2}\Phi^L\overline{\Phi^L}\right) , \qquad g_{I\bar{J}}(\Phi, \bar{\Phi}) = \frac{r^2\delta_{IJ}}{r^2 + \Phi^L\overline{\Phi^L}} - \frac{r^2\overline{\Phi^I}\Phi^J}{(r^2 + \Phi^L\overline{\Phi^L})^2} , \tag{9.9}$$

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[21]In what follows, we do not indicate the bar-projection.

where $I, \bar{J} = 1, \ldots, n$ and $r^2 = $ const. We recall that the Riemann curvature of $\mathbb{C}P^n$ is

$$R_{I_1 \bar{J}_1 I_2 \bar{J}_2} := K_{I_1 \bar{J}_1 I_2 \bar{J}_2} - g_{M\bar{N}} \Gamma^M_{I_1 I_2} \bar{\Gamma}^{\bar{N}}_{\bar{J}_1 \bar{J}_2} = -\frac{1}{r^2} \left\{ g_{I_1 \bar{J}_1} g_{I_2 \bar{J}_2} + g_{I_1 \bar{J}_2} g_{I_2 \bar{J}_1} \right\} , \quad (9.10)$$

and hence

$$\Sigma^{I_1} \bar{\Sigma}^{\bar{J}_1} \Sigma^{I_2} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = -\frac{2}{r^2} g_{I_2 \bar{J}_2} \Sigma^{I_2} |\Sigma|^2 , \qquad |\Sigma|^2 := g_{I\bar{J}}(\Phi, \bar{\Phi}) \, \Sigma^I \bar{\Sigma}^{\bar{J}} . \quad (9.11)$$

Upon elimination of the auxiliary superfields, the Lagrangian becomes[22] [59]

$$\mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = G \, K(\Phi, \bar{\Phi}) + \chi K_I(\Phi, \bar{\Phi}) \Sigma^I + \bar{\chi} K_{\bar{J}}(\Phi, \bar{\Phi}) \bar{\Sigma}^{\bar{J}}$$
$$+ r^2 \left\{ G \ln \frac{1 - |\Sigma|^2/r^2}{\sqrt{G^2 + 4\bar{\chi}\chi(1 - |\Sigma|^2/r^2)} + G} + \sqrt{G^2 + 4\bar{\chi}\chi(1 - |\Sigma|^2/r^2)} \right\}$$
$$- r^2 \left\{ \mathbb{H} - G \ln \left(G + \mathbb{H} \right) \right\} , \quad (9.12)$$

where

$$\mathbb{H} := \sqrt{G^2 + 4\chi\bar{\chi}} . \quad (9.13)$$

The theory (9.12) possesses a dual formulation obtained by dualizing the complex linear tangent variables Σ^I and their conjugates $\bar{\Sigma}^{\bar{I}}$ into chiral superfields Ψ_I and their conjugates $\bar{\Psi}_{\bar{I}}$, $\bar{\mathcal{D}}_{\dot{\alpha}}\Psi_I = 0$. One first replaces the action with a first order one,

$$S = \int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \left\{ \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Sigma^I \Psi_I + \bar{\Sigma}^{\bar{J}} \bar{\Psi}_{\bar{J}} \right\} , \quad (9.14)$$

where Σ^I and $\bar{\Sigma}^{\bar{J}}$ are chosen to be complex unconstrained. Next, one eliminates these superfields with the aid of their algebraic equations of motions, ending up with the dual Lagrangian [59]:

$$\mathbf{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = G \, K(\Phi, \bar{\Phi}) - r^2 \left\{ \mathbb{H} - G \ln \left(G + \mathbb{H} \right) \right\}$$
$$+ r^2 \left\{ \sqrt{\mathbb{H}^2 + 4|\Psi + \chi\nabla K|^2/r^2} - G \ln \left(\sqrt{\mathbb{H}^2 + 4|\Psi + \chi\nabla K|^2/r^2} + G \right) \right\} , \quad (9.15)$$

where

$$|\Psi + \chi\nabla K|^2 := g^{I\bar{J}} \left(\Psi_I + \chi K_I(\Phi, \bar{\Phi}) \right) \left(\bar{\Psi}_{\bar{J}} + \bar{\chi} K_{\bar{J}}(\Phi, \bar{\Phi}) \right) . \quad (9.16)$$

Under the Kähler transformation (1.27), the chiral one-form Ψ_I changes as

$$\Psi_I \quad \longrightarrow \quad \Psi_I - \chi \, F_I(\Phi) , \quad (9.17)$$

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[22]We view the tensor multiplet as a background field.

and this transformation is clearly consistent with the chirality of Ψ_I. The reason for the non-invariance of Ψ_I is that the terms

$$\int \mathrm{d}^4 x \, \mathrm{d}^2\theta \, \mathrm{d}^2\bar{\theta} \, E \left\{ \chi K_I(\Phi, \bar{\Phi})\Sigma^I + \bar{\chi} K_{\bar{J}}(\Phi, \bar{\Phi})\bar{\Sigma}^{\bar{J}} \right\} \tag{9.18}$$

in (9.14) are not invariant under the Kähler transformations when Σ^I is complex unconstrained.

In the limit $G = 1$ and $\chi = 0$,[23] the Lagrangian (9.15) reduces to the standard hyperkähler potential for the cotangent bundle of $\mathbb{C}P^n$, see e.g. [21],

$$\mathbb{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = K(\Phi, \bar{\Phi}) + r^2 \left(\sqrt{1 + 4|\Psi|^2/r^2} - 1 \right) - r^2 \ln \frac{\sqrt{1 + 4|\Psi|^2/r^2} + 1}{2} \ . \tag{9.19}$$

This is the hyperkähler potential appearing in the 3D foliated action (6.26) with Darboux coordinates Φ^I and Ψ_I. It is invariant under a U(1) Killing vector field V of the standard form (7.12b). The complex coordinates Φ^I and Ψ_I naturally diagonalize the preferred complex structure $\mathbb{J} = \mathcal{J}_3$ with respect to which V is holomorphic.

We are actually interested in a different limit, $G = 0$ and $\chi = \mathrm{i}\mu/2|\mu|$, which gives us the formulation in terms of $\mathcal{N} = 1$ chiral superfields for the σ-model (1.26). Implementing this limit gives

$$\mathcal{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = r^2 \sqrt{1 + \frac{4}{r^2}\left|\Psi + \mathrm{i}\frac{\mu}{2|\mu|}\nabla K(\Phi, \bar{\Phi})\right|^2} - r^2 \ , \tag{9.20}$$

with a *different* set of Darboux coordinates Φ^I and Ψ_I, which diagonalize a different complex structure $J_3 = \mathcal{J}_1$. The Lagrangian is globally defined on $T^*\mathbb{C}P^n$. The corresponding U(1) Killing vector field (1.13) is not holomorphic with respect to the diagonalized $J_3 = \mathcal{J}_1$, but rather with respect to $\mathbb{J} = \mathcal{J}_3$. This should be compared with eq. (7.18) which is the expression for $\mathcal{K}$ in terms of the complex coordinates which diagonalize $\mathbb{J} = \mathcal{J}_3$.

10 Conclusions

One of the important findings of this work is the observation that the most general $\mathcal{N} = 2$ supersymmetric σ-model in AdS_4 can be described in terms of an off-shell σ-model

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[23]It should be kept in mind that the limit $G = 1$ and $\chi = 0$ cannot be performed in the AdS frame, but instead only in the 3D foliated frame.

in projective superspace given by eqs. (1.23) and (1.26). This model is associated with a real analytic Kähler manifold $\mathcal{X}$ with Kähler potential $K(\Phi^I, \bar{\Phi}^{\bar{J}})$ which appears in (1.26). As demonstrated above, there are two ways to relate this off-shell formulation, realized in terms of covariant weight-zero arctic multiplets, to a formulation in terms of ordinary chiral superfields: (i) using the AdS frame; and (ii) using the 3D foliated frame. In the AdS frame, one ends up with the σ-model (1.9) in which the Lagrangian $\mathcal{K}(\varphi^a, \bar{\varphi}^{\bar{b}})$ is a globally defined function over the hyperkähler target space $\mathcal{M}$ such that $\mathcal{K}$ is the Kähler potential, $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} \mathcal{K}$, with respect to any complex structure orthogonal to the preferred one $\mathbb{J}$, eq. (1.16), which is invariant under the Killing vector V rotating the complex structure. The covariantly chiral superfields φ^a in (1.9) are complex coordinates with respect to a certain complex structure, J_3, orthogonal to $\mathbb{J}$. In the 3D foliated frame, one ends up with the σ-model (6.26) in which the Lagrangian is

$$\mathbb{K}(\phi^{\mathrm{a}}, \bar{\phi}^{\bar{\mathrm{b}}}) \equiv \mathbb{K}(\Phi^I, \Psi_I, \bar{\Phi}^{\bar{J}}, \bar{\Psi}_{\bar{J}}) = K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) , \qquad (10.1)$$

where

$$\mathcal{H}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \ldots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \ldots \bar{\Psi}_{\bar{J}_n} . \qquad (10.2)$$

Here $\mathcal{H}^{I\bar{J}} = g^{I\bar{J}}$ and the coefficients $\mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}$, for $n > 1$, are tensor functions of the Kähler metric $g_{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}} K(\Phi, \bar{\Phi})$ on $\mathcal{X}$, the Riemann curvature $R_{I\bar{J}K\bar{L}}(\Phi, \bar{\Phi})$ and its covariant derivatives. The superfield Lagrangian $\mathbb{K}(\phi^{\mathrm{a}}, \bar{\phi}^{\bar{\mathrm{b}}})$ is the Kähler potential of $\mathcal{M}$ in complex coordinates ϕ^{a} with respect to the preferred complex structure $\mathbb{J}$. Associated with $\mathbb{K}(\phi^{\mathrm{a}}, \bar{\phi}^{\bar{\mathrm{b}}})$ is the globally defined function of $\mathcal{M}$

$$\mathcal{K}(\phi^{\mathrm{a}}, \bar{\phi}^{\bar{\mathrm{b}}}) = 2\Psi_I \frac{\partial}{\partial \Psi_I} \mathcal{H}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) , \qquad (10.3)$$

which is the Killing potential for the Killing vector V (which is holomorphic with respect to $\mathbb{J}$). The function $\mathcal{K}(\phi^{\mathrm{a}}, \bar{\phi}^{\bar{\mathrm{b}}})$ coincides with the superfield Lagrangian $\mathcal{K}(\varphi^a, \bar{\varphi}^{\bar{b}})$ in (1.9), however they are written down in terms of different coordinates for $\mathcal{M}$. The former is given in terms of the complex coordinates with respect to $\mathbb{J}$, while the latter is defined in terms of the variables φ^a which are complex coordinates with respect to the orthogonal complex structure J_3.

In conjunction with the results of [1, 2], our work shows that there is a one-to-one correspondence between $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ and those hyperkähler manifolds which possess a Killing vector field generating an SO(2) group of rotations on the two-sphere of complex structures. This clearly differs from $\mathcal{N} = 2$ Poincaré supersymmetry where arbitrary hyperkähler manifolds can originate as target spaces of $\mathcal{N} = 2$

supersymmetric σ-models [3, 45]. The difference between the σ-model target spaces which are allowed by $\mathcal{N} = 2$ Poincaré and AdS supersymmetries can nicely be demonstrated in terms of the most general off-shell 4D $\mathcal{N} = 2$ supersymmetric σ-model in *flat* projective superspace $\mathbb{R}^{4|8} \times \mathbb{C}P^1$ formulated in [8]. The action is

$$ S = \oint_C \frac{d\zeta}{2\pi i\zeta} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, L(\Upsilon^I, \check{\Upsilon}^{\bar{J}}, \zeta) \,, \tag{10.4} $$

where the Lagrangian is an essentially arbitrary function of its arguments. As shown in [60], $L(\Upsilon^I, \check{\Upsilon}^{\bar{J}}, \zeta)$ has a geometric origin and can be defined for any hyperkähler manifold. The target space of this $\mathcal{N} = 2$ supersymmetric σ-model in Minkowski space can at the same time originate as the target space of some $\mathcal{N} = 2$ supersymmetric σ-model in AdS$_4$ only if the Lagrangian has no explicit ζ-dependence,

$$ L(\Upsilon^I, \check{\Upsilon}^{\bar{J}}, \zeta) \;\rightarrow\; K(\Upsilon^I, \check{\Upsilon}^{\bar{J}}) \,. \tag{10.5} $$

In the case of Minkowski space, it is well known [61, 45] that adding a superpotential to an $\mathcal{N} = 2$ supersymmetric σ-model requires the target space to possess a tri-holomorphic Killing vector field. We have found an additional restriction in the AdS case: this tri-holomorphic Killing vector must commute with the Killing vector V which rotates the complex structures.

Many results of our work can be naturally extended to five dimensions. Within the projective-superspace setting, general off-shell σ-models in 5D $\mathcal{N} = 1$ AdS superspace were formulated in [15]. A 5D analogue of the 3D foliated frame was developed in [16]. One can repeat the analysis of section 6 for the case of the off-shell $\mathcal{N} = 1$ σ-models in AdS$_5$ proposed in [15]. The results of such an analysis will be the most general 5D $\mathcal{N} = 1$ supersymmetric σ-model in AdS$_5$ realized in terms of 4D $\mathcal{N} = 1$ chiral superfields [62, 63].

There still remain a number of interesting open questions. In particular, in our discussion of gauged $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$ the vector multiplet was chosen to be intrinsic, since our goal was to derive the superpotential generated. It is of interest to study the general structure of gauged $\mathcal{N} = 2$ supersymmetric σ-models in AdS$_4$. This will be reported in a separate publication [64].

Another interesting issue, which we only briefly touched upon in section 7.1, was that the supersymmetry algebra of the general 3D foliated σ-model closes off-shell, which is quite distinct from the Minkowski case [45]. It was shown in [1, 2] that the general $\mathcal{N} = 2$ supersymmetric σ-model in the AdS frame also has a closed algebra, with the SO(2) generator of AdS mimicking the action of a central charge. It is unsurprising that the 3D

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foliation should have the same feature, and it would be interesting to develop an off-shell Fayet-Sohnius $\mathcal{N} = 2$ superfield formulation (as in the AdS frame [2]) for the 3D foliated frame.

One last question regards the two choices of s^{ij} we have made, eq. (1.3b), which led to the AdS and 3D foliated frames. At the level of the hyperkähler target space, these two frames are related by a non-holomorphic coordinate transformation which effects a rotation on the complex structures. At the same time, this coordinate transformation acts as a simple SU(2) rotation on the original projective multiplets which defined the action. The explicit link between these two operations remains unexplored. We expect that for the wide class of symmetric spaces studied in [18, 19, 20, 21, 22, 23], it should be possible to construct this coordinate transformation explicitly.

A Killing vectors

Within the formulation of $\mathcal{N} = 2$ conformal supergravity used in this paper [11], a superconformal Killing vector consists of a superspace diffeomorphism and structure group transformation encoded in the parameter

$$\xi = \xi^{\mathcal{A}}\mathcal{D}_{\mathcal{A}} + \frac{1}{2}\lambda^{cd}M_{cd} + \lambda^{ij}J_{ij} \ , \tag{A.1}$$

along with a super-Weyl transformation[24] associated with a chiral superfield Σ, so that the covariant derivatives are invariant,

$$\delta_\xi \mathcal{D}_\mathcal{A} + \delta_\Sigma \mathcal{D}_\mathcal{A} = 0 \ . \tag{A.2}$$

If the covariant derivatives $\mathcal{D}_\mathcal{A}$ are associated with an AdS geometry, then the AdS Killing vectors are those with $\Sigma = 0$, i.e. $\delta_\xi \mathcal{D}_\mathcal{A} = 0$. For such a geometry, the properties that the Killing vector ξ must obey have been worked out in detail [10]. For our purposes, the relevant features are that all the parameters can be derived from $\xi_{\alpha\dot\alpha}$, which must obey the relations

$$\mathcal{D}^i_{(\beta}\xi_{\alpha)\dot\alpha} = \bar{\mathcal{D}}^{(\dot\beta}_i \xi^{\dot\alpha)\alpha} = 0 \ , \tag{A.3a}$$

$$\mathcal{D}^j_\beta \bar{\mathcal{D}}_{\dot\beta j} \xi^{\dot\beta\beta} = \bar{\mathcal{D}}_{\dot\beta j} \mathcal{D}^j_\beta \xi^{\dot\beta\beta} = 0 \ . \tag{A.3b}$$

The first condition, known as the master equation, holds for a superconformal Killing vector, while the second imposes the additional requirement that the vector is AdS Killing. Together, it is easy to show that these imply the superspace version of the usual Killing equation

$$\mathcal{D}_a \xi_b + \mathcal{D}_b \xi_a = 0 \ . \tag{A.4}$$

The remaining parameters ξ^α_i, λ^{ab} and λ^{ij} can be derived from ξ_a and are given in [10] (see also [2]). For example, one finds that $\lambda^{ij} = 2\varepsilon \mathcal{S}^{ij}$ for some real superfield ε.

Our interest is in the 3D foliated version of AdS, which we constructed explicitly in section 4.2 in terms of a chiral superfield σ given in (4.39) with the choice (5.25) and $\alpha = -i$. In section A.1, we explicitly construct a solution for the AdS Killing vectors using σ. Then in section A.2, we perform the rotation described in section 5.2 and give a new form of the Killing vectors relevant for the 3D $\mathcal{N} = 2$ superspace used in sections 6 and 7.

A.1 Killing vectors in 3D foliated AdS

As discussed in section 4.2, the flat Minkowski derivatives $D_\mathcal{A}$ are related to the AdS covariant derivatives $\mathcal{D}_\mathcal{A}$ by a super-Weyl transformation. Superconformal Killing vectors on both spaces are similarly related. The easiest way to derive the relation is to consider

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[24]Within [11], the super-Weyl parameter was denoted σ. Here we use Σ to avoid confusion with the finite super-Weyl transformation σ connecting the AdS frame to the Minkowski frame.

a superconformal Killing isometry acting on a scalar function $\mathcal{F}$ of vanishing super-Weyl weight. Within AdS, this isometry is

$$\delta\mathcal{F} = -\xi^{\mathcal{A}}\mathcal{D}_{\mathcal{A}} = -\xi^{\mathcal{A}}E_{\mathcal{A}}\mathcal{F} \tag{A.5}$$

while in the Minkowski frame

$$\delta\mathcal{F} = -\widetilde{\xi}^{\mathcal{A}}D_{\mathcal{A}}\mathcal{F} \ . \tag{A.6}$$

Equating the two results, one easily finds

$$\widetilde{\xi}_{\alpha\dot\alpha} = \xi_{\alpha\dot\alpha}\,\mathrm{e}^{(\sigma+\bar\sigma)/2} \ , \qquad \widetilde{\xi}_i^{\alpha} = \xi_i^{\alpha}e^{\bar\sigma/2} + e^{(\sigma+\bar\sigma)/2}\left(\frac{\mathrm{i}}{4}\xi^{\alpha}{}_{\dot\beta}\bar{D}_i^{\dot\beta}\bar\sigma\right) \tag{A.7}$$

where σ is the super-Weyl transformation connecting the AdS frame to the Minkowski frame. One can show that if $\xi_{\mathcal{A}}$ is a superconformal Killing vector in AdS, $\widetilde{\xi}_{\mathcal{A}}$ must be a superconformal Killing vector in Minkowski, and vice-versa.

Now we specialize to the case where $\xi_{\mathcal{A}}$ is an AdS Killing vector, obeying both equations (A.3). One can show that $\widetilde{\xi}_{\mathcal{A}}$ obeys

$$D_{(\beta}^i\widetilde{\xi}_{\alpha)\dot\alpha} = \bar{D}_i^{(\dot\beta}\widetilde{\xi}^{\dot\alpha)\alpha} = 0 \ , \tag{A.8a}$$

$$D_{\beta}^j\bar{D}_{\dot\beta j}\widetilde{\xi}^{\dot\beta\beta} = 16\mathrm{i}\partial_a\bar\sigma\widetilde{\xi}^a - 2\bar{D}_{\dot j}^{\dot\alpha}\bar\sigma D^{\alpha j}\widetilde{\xi}_{\alpha\dot\alpha} \ . \tag{A.8b}$$

Given these equations, it is straightforward to construct explicitly the solution for $\widetilde{\xi}_{\alpha\dot\alpha}$, from which all the other parameters can be constructed. The general solution to the first equation (A.8a) is the general $\mathcal{N}=2$ superconformal Killing vector in Minkowski, and is given by

$$\widetilde{\xi}_i^{\alpha} = \epsilon_i^{\alpha} + \bar{r}\theta_i^a - \theta_i^{\beta}\omega_{\beta}{}^{\alpha} - \Lambda_i{}^j\theta_j^{\alpha} + \theta_i^{\beta}k_{\beta\dot\beta}x_L^{\dot\beta\alpha} - \mathrm{i}\bar\eta_{i\dot\beta}x_L^{\dot\beta\alpha} - 4\theta_i^{\beta}\eta_{\beta}^k\theta_k^{\alpha} \tag{A.9a}$$

$$\widetilde{\xi}^a = \frac{1}{2}(\widetilde{\xi}_L^a + \widetilde{\xi}_R^a) - \mathrm{i}\widetilde{\xi}_k\sigma^a\bar\theta^k - \mathrm{i}\widetilde{\bar\xi}^k\tilde\sigma^a\theta_k \tag{A.9b}$$

$$\widetilde{\xi}_L^{\dot\alpha\alpha} = p^{\dot\alpha\alpha} + (r+\bar{r})x_L^{\dot\alpha\alpha} - \bar\omega^{\dot\alpha}{}_{\dot\beta}x_L^{\dot\beta\alpha} - x_L^{\dot\alpha\beta}\omega_{\beta}{}^{\alpha} + x_L^{\dot\alpha\beta}k_{\beta\dot\beta}x_L^{\dot\beta\alpha} + 4\mathrm{i}\bar\epsilon^{\dot\alpha k}\theta_k^{\alpha} - 4x_L^{\dot\alpha\beta}\eta_{\beta}^k\theta_k^{\alpha} \tag{A.9c}$$

where $x_L^a := x^a + \mathrm{i}\theta_j\sigma^a\bar\theta^j$. The last equation above can be rewritten

$$\widetilde{\xi}_L^a = p^a + (r+\bar{r})x_L^a + \omega^a{}_bx_L^b - 2x_L^bk_bx_L^a + x_L^2k^a - 2\mathrm{i}\bar\epsilon^k\tilde\sigma^a\theta_k - 2x_L^b\eta^k\sigma_b\tilde\sigma^a\theta_k \ . \tag{A.10}$$

The constant parameters ϵ_i^{α} and p_a are the supersymmetry and spacetime translation parameters, $\omega^a{}_b$ is the constant Lorentz parameter, and $\Lambda_i{}^j$ is the SU(2) parameter. The real and imaginary parts of r give constant dilatation and U(1) transformations. Finally, k_a and η_{α}^i are the special conformal and S-supersymmetry parameters.

The second condition (A.8b) imposes restrictions on some of these constant parameters. One finds for the bosonic parameters

$$p_z = 0 \;, \qquad \omega^z{}_b = 0 \;, \qquad \Lambda^{\underline{11}} = \Lambda^{\underline{22}} = 0 \;, \qquad r = \bar{r} \;, \qquad k_z = 0 \;. \tag{A.11}$$

The first two constraints eliminate translations in the z direction as well as rotations mixing z with the 3D coordinates. The third constraint arises from $\Lambda^{ij} \propto s^{ij}$ as usual in AdS. The fourth eliminates the global U(1) freedom while the last eliminates special conformal transformations in the z direction. The constraints we find on the fermionic parameters are[25]

$$\epsilon_{\alpha\underline{1}} = \mathrm{i}\bar{\epsilon}_\alpha^{\underline{2}} \;, \qquad \epsilon_{\alpha\underline{2}} = \mathrm{i}\bar{\epsilon}_\alpha^{\underline{1}} \;, \qquad \eta^{\alpha\underline{1}} = \mathrm{i}\bar{\eta}_{\underline{2}}^\alpha \;, \qquad \eta^{\alpha\underline{2}} = \mathrm{i}\bar{\eta}_{\underline{1}}^\alpha \;. \tag{A.12}$$

In addition to the parameters $\widetilde{\xi}^A$, we will also need the explicit form of the SU(2) parameter $\widetilde{\lambda}_{ij}$, which is given by

$$\widetilde{\lambda}_{ij} = \frac{1}{2} D_{\beta(i}\widetilde{\xi}_{j)}^\beta = -\Lambda_{ij} - 4\theta_{(i}\eta_{j)} + 4\bar{\theta}_{(i}\bar{\eta}_{j)} + 4\mathrm{i}k_{\hat{b}}\theta_{(i}\sigma^{\hat{b}}\bar{\theta}_{j)} \;. \tag{A.13}$$

One could easily derive the Lorentz parameter $\widetilde{\lambda}^{ab}$, but we have no need for it.

A.2   A new basis for the Killing vectors

In section 5.2, we introduced a new basis for the spinor covariant derivatives. The Killing vectors in the new basis, which we denote $\boldsymbol{\xi}^{\mathcal{A}}$, can be found by identifying

$$\widetilde{\xi}^a\partial_a + \widetilde{\xi}_i^\alpha D_\alpha^i + \bar{\widetilde{\xi}}_{\dot{\alpha}}^i \bar{D}_i^{\dot{\alpha}} = \boldsymbol{\xi}^{\hat{a}}\partial_{\hat{a}} + \boldsymbol{\xi}^z\partial_z + \boldsymbol{\xi}_i^\alpha \boldsymbol{D}_\alpha^i + \bar{\boldsymbol{\xi}}_\alpha^i \bar{\boldsymbol{D}}_i^\alpha \tag{A.14}$$

where the index $\hat{a}$ is restricted now to run over the 3D coordinates. A direct identification of these two formulae leads to[26]

$$\boldsymbol{\xi}^{\hat{a}} := \widetilde{\xi}^{\hat{a}} \;, \quad \boldsymbol{\xi}^z := \widetilde{\xi}^2 \;, \quad \boldsymbol{\xi}_i^\alpha := \frac{1}{\sqrt{2}}\left(\widetilde{\xi}_i^\alpha - \mathrm{i}\bar{\widetilde{\xi}}_i^\alpha\right) \;, \quad \bar{\boldsymbol{\xi}}_\alpha^i := \frac{1}{\sqrt{2}}\left(\bar{\widetilde{\xi}}_\alpha^i - \mathrm{i}\widetilde{\xi}_\alpha^i\right) \;. \tag{A.15}$$

In constructing the Killing vectors in this rotated basis, we should also rotate the constant fermionic parameters. Defining

$$\boldsymbol{\epsilon}_i^\alpha := \frac{1}{\sqrt{2}}(\epsilon_i^\alpha - \mathrm{i}\bar{\epsilon}_i^\alpha) \;, \qquad \boldsymbol{\eta}^{\alpha i} := \frac{1}{\sqrt{2}}(\eta^{\alpha i} + \mathrm{i}\bar{\eta}^{\alpha i}) \;, \tag{A.16}$$

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[25]Here we make the choice $\alpha = -\mathrm{i}$ as in the main body of the paper.
[26]We take $\lambda = \mathrm{i}$ as in the main body of the paper.

we find that the constraints (A.12) amount to

$$\epsilon_{\underline{2}}^{\alpha} = 0 \,, \qquad \boldsymbol{\eta}^{\alpha \underline{2}} = 0 \,. \tag{A.17}$$

In this transformed basis, a projective multiplet of weight zero transforms as

$$\delta \mathcal{Q} = - \left(\boldsymbol{\xi}^{\hat{a}} \partial_{\hat{a}} + \boldsymbol{\xi}^{z} \partial_{z} + \boldsymbol{\xi}_{i}^{\alpha} \boldsymbol{D}_{\alpha}^{i} + \bar{\boldsymbol{\xi}}_{\alpha}^{i} \bar{\boldsymbol{D}}_{i}^{\alpha} + \left(\widetilde{\lambda}_{\underline{11}} + 2 \widetilde{\lambda}_{\underline{12}} \zeta + \widetilde{\lambda}_{\underline{22}} \zeta^{2} \right) \partial_{\zeta} \right) \mathcal{Q} \,. \tag{A.18}$$

It is possible to work out the new Killing vectors explicitly, but we need only their projection to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^{2} = 0$. It turns out that $\widetilde{\lambda}_{\underline{11}}| = \widetilde{\lambda}_{\underline{22}}| = 0$. The remaining pieces are defined as

$$\xi^{\hat{a}} := \boldsymbol{\xi}^{\hat{a}}| \,, \qquad \xi^{z} := \boldsymbol{\xi}^{z}| \,, \qquad \xi^{\alpha} := \boldsymbol{\xi}_{\underline{1}}^{\alpha}| \,, \qquad \rho^{\alpha} := \boldsymbol{\xi}_{\underline{2}}^{\alpha}| \,, \qquad \Lambda := \mathrm{i} \widetilde{\lambda}_{\underline{12}}| \,. \tag{A.19}$$

We also use abbreviated notation for the constant parameters and for the residual $\boldsymbol{\theta}_{\underline{1}}$ and $\bar{\boldsymbol{\theta}}^{\underline{1}}$ coordinates:

$$\epsilon^{\alpha} := \epsilon_{\underline{1}}^{\alpha} \,, \qquad \eta^{\alpha} := \boldsymbol{\eta}^{\alpha \underline{1}} \,, \qquad \boldsymbol{\theta} = \boldsymbol{\theta}_{\underline{1}} \,. \tag{A.20}$$

In terms of these, one finds[27]

$$\begin{aligned}
\xi^{\hat{a}} &= p^{\hat{a}} + \omega^{\hat{a}}{}_{\hat{b}} x^{\hat{b}} - \frac{\mathrm{i}}{2} \varepsilon^{\hat{a}\hat{b}\hat{c}} \omega_{\hat{b}\hat{c}} \boldsymbol{\theta}\bar{\boldsymbol{\theta}} - 2\mathrm{i}\,\bar{\epsilon}\tilde{\sigma}^{\hat{a}}\boldsymbol{\theta} - 2\mathrm{i}\,\epsilon\sigma^{\hat{a}}\bar{\boldsymbol{\theta}} + 2rx^{\hat{a}} + 2\mathrm{i}\Lambda_{\underline{1}}{}^{\underline{1}}\boldsymbol{\theta}\sigma^{\hat{a}}\bar{\boldsymbol{\theta}} \\
&\quad - 2x^{\hat{b}} k_{\hat{b}} x^{\hat{a}} + x^{2} k^{\hat{a}} - 2\mathrm{i}\varepsilon^{\hat{a}\hat{b}\hat{c}} k_{\hat{b}} x_{\hat{c}} \boldsymbol{\theta}\bar{\boldsymbol{\theta}} + z^{2} k^{\hat{a}} - \frac{1}{2}\boldsymbol{\theta}^{2}\bar{\boldsymbol{\theta}}^{2} k^{\hat{a}} \\
&\quad - 2x^{\hat{b}}(\eta\sigma_{\hat{b}}\tilde{\sigma}^{\hat{a}}\boldsymbol{\theta}) - 2x^{\hat{b}}(\bar{\eta}\tilde{\sigma}_{\hat{b}}\sigma^{\hat{a}}\bar{\boldsymbol{\theta}}) - \mathrm{i}\,(\eta\sigma^{\hat{a}}\bar{\boldsymbol{\theta}})\boldsymbol{\theta}^{2} + \mathrm{i}\,(\boldsymbol{\theta}\sigma^{\hat{a}}\bar{\eta})\bar{\boldsymbol{\theta}}^{2} \,, \tag{A.21a} \\
\xi^{\alpha} &= \epsilon^{\alpha} - \frac{1}{2}\omega^{\hat{b}\hat{c}}(\boldsymbol{\theta}\sigma_{\hat{b}\hat{c}})^{\alpha} - \Lambda_{\underline{1}}{}^{\underline{1}}\boldsymbol{\theta}^{\alpha} + k^{\hat{b}} x^{\hat{c}}(\boldsymbol{\theta}\sigma_{\hat{b}}\tilde{\sigma}_{\hat{c}})^{\alpha} - \frac{\mathrm{i}}{2}k_{\hat{b}}\boldsymbol{\theta}^{2}\bar{\boldsymbol{\theta}}^{\beta}(\sigma^{\hat{b}})_{\beta}{}^{\alpha} \\
&\quad - \mathrm{i} x^{\hat{b}}\bar{\boldsymbol{\eta}}_{\beta}(\tilde{\sigma}_{\hat{b}})^{\beta\alpha} + (\boldsymbol{\theta}\bar{\boldsymbol{\theta}})\bar{\eta}^{\alpha} + \eta^{\alpha}\boldsymbol{\theta}^{2} + 2\boldsymbol{\theta}^{\beta}\bar{\eta}_{\beta}\bar{\boldsymbol{\theta}}^{\alpha} \,, \tag{A.21b} \\
\xi^{z} &= 2rz - 2x^{\hat{b}} k_{\hat{b}} z + 2(\eta\boldsymbol{\theta} + \bar{\eta}\bar{\boldsymbol{\theta}})z \,, \tag{A.21c} \\
\rho^{\alpha} &= \mathrm{i} z \eta^{\alpha} + z k_{\hat{b}}\bar{\boldsymbol{\theta}}^{\beta}(\sigma^{\hat{b}})_{\beta}{}^{\alpha} \,, \tag{A.21d} \\
\Lambda &= -\mathrm{i}\Lambda_{\underline{1}}{}^{\underline{1}} - 2\mathrm{i}\eta\boldsymbol{\theta} + 2\mathrm{i}\bar{\eta}\bar{\boldsymbol{\theta}} - 2k_{\hat{a}}\boldsymbol{\theta}\sigma^{\hat{a}}\bar{\boldsymbol{\theta}} \,. \tag{A.21e}
\end{aligned}$$

Taking the $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^{2} = 0$ projection of (A.18) gives the transformations (6.1).

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[27] The 4D sigma matrices used here can be replaced with 3D gamma matrices. The result is given in (6.2) and (6.3).

B Prepotential for the intrinsic vector multiplet

An abelian vector multiplet is described in Minkowski superspace by a reduced chiral superfield W obeying

$$\bar{D}_i^{\dot{\alpha}} W = D_\alpha^i \bar{W} = 0 \ , \quad D^{ij} W = \bar{D}^{ij} \bar{W} \ . \tag{B.1}$$

Within projective superspace, these constraints are solved in terms of a projective prepotential V, which is a real weight zero tropical superfield

$$V(z,v) = \sum_{k=-\infty}^{+\infty} \zeta^k V_k(z) \ , \quad \bar{V}_k = (-1)^k V_{-k} \ , \quad D_\alpha^{(1)} V = \bar{D}_\alpha^{(1)} V = 0 \ , \tag{B.2}$$

so that

$$W = -\frac{1}{8\pi} \oint \frac{(v, \mathrm{d}v)}{(v,u)^2} \bar{D}^{(-2)} V \ , \qquad \bar{W} = -\frac{1}{8\pi} \oint \frac{(v, \mathrm{d}v)}{(v,u)^2} D^{(-2)} V \ . \tag{B.3}$$

Here we have introduced the operators

$$D_\alpha^{(-1)} := \frac{1}{(v,u)} u_i D_\alpha^i \ , \quad \bar{D}_{\dot{\alpha}}^{(-1)} := \frac{1}{(v,u)} u_i \bar{D}_\alpha^i \ , \tag{B.4a}$$

$$D^{(-2)} := D^{\alpha(-1)} D_\alpha^{(-1)} \ , \quad \bar{D}^{(-2)} := \bar{D}_{\dot{\alpha}}^{(-1)} \bar{D}^{\dot{\alpha}(-1)} \ . \tag{B.4b}$$

The field strength W is invariant under the gauge transformations

$$V \to V + \mathrm{i}\Lambda - \mathrm{i}\check{\Lambda} \tag{B.5}$$

for a weight zero arctic superfield Λ.

Let us now specialize to the AdS intrinsic vector multiplet. For the case $\boldsymbol{s}^{\underline{11}} = \boldsymbol{s}^{\underline{22}} = 0$ and $\boldsymbol{s}^{\underline{12}} = \alpha s$, with $\alpha = \pm \mathrm{i}$, we have

$$W = \frac{1}{s} \frac{1}{(z_L + 2\alpha\theta_{\underline{12}})} \ . \tag{B.6}$$

In the north chart of $\mathbb{C}P^1$, the prepotential V can be chosen

$$V = \frac{1}{s z_A(\zeta)} \frac{\mathrm{i}}{\zeta} \left(\theta^{(2)}(\zeta) + \bar{\theta}^{(2)}(\zeta) \right) - \frac{1}{s(z_A(\zeta))^2} \frac{\mathrm{i}\alpha}{\zeta^2} \theta^{(2)}(\zeta) \bar{\theta}^{(2)}(\zeta) \ , \tag{B.7}$$

where the analytic coordinates in the north chart are

$$\theta_\alpha^{(1)}(\zeta) := \zeta_i \theta_\alpha^i \ , \qquad \bar{\theta}_\alpha^{(1)}(\zeta) := \zeta_i \bar{\theta}_\alpha^i \ , \quad \zeta^i := (1, \zeta) \ , \tag{B.8a}$$

$$\theta^{(2)}(\zeta) = \theta^{\alpha(1)}(\zeta) \theta_\alpha^{(1)}(\zeta) \ , \qquad \bar{\theta}^{(2)}(\zeta) = \bar{\theta}_\alpha^{(1)}(\zeta) \bar{\theta}^{\alpha(1)}(\zeta) \ , \tag{B.8b}$$

$$z_A(\zeta) = z + \theta^{\alpha(1)}(\zeta) \bar{\theta}_\alpha^{\underline{1}} + \theta^{\alpha\underline{1}} \bar{\theta}_\alpha^{(1)}(\zeta) \ . \tag{B.8c}$$

One can check that this expression for V is indeed real under smile conjugation. Upon reduction to 3D $\mathcal{N} = 2$ superspace, we find

$$V\big| = \frac{\mathrm{i}}{sz}\left(\frac{1}{\zeta}\boldsymbol{\theta}^2 + \zeta\bar{\boldsymbol{\theta}}^2\right) . \tag{B.9}$$

This prepotential corresponds to a frozen vector multiplet, which necessarily has vanishing component field strength F_{mn}. The specific choice of V made above turns out to correspond to a vanishing component gauge connection A_m as well. It remains possible to perform a gauge transformation on V to yield a non-vanishing (but pure gauge) A_m. We will demonstrate this for the special case of an arbitrary z-dependent gauge connection A_z.

Let us perform a gauge transformation with $\Lambda = \Lambda(z_A)$ where Λ is a real function and

$$z_A := z + \theta^{(1)}\bar{\theta}^{(-1)} + \theta^{(-1)}\bar{\theta}^{(1)} , \tag{B.10a}$$

$$\theta^{(-1)}_\alpha := \frac{\theta^i u_i}{\zeta^i u_i} , \qquad \bar{\theta}^{(-1)}_\alpha := \frac{\bar{\theta}^i u_i}{\zeta^i u_i} , \qquad u_i = (1, 0) . \tag{B.10b}$$

The quantity z_A has already been introduced, and is well-defined in the north chart. Its smile conjugate $\check{z}_A$ is given by

$$\check{z}_A = z + \theta^{(1)}\bar{\theta}'^{(-1)} + \theta'^{(-1)}\bar{\theta}^{(1)} , \quad u'_i = (0, 1) , \tag{B.11}$$

and is well-defined in the south chart. The antarctic gauge parameter is then given by $\check{\Lambda} = \Lambda(\check{z}_A)$. Now rotating the theta coordinates and projecting to the 3D $\mathcal{N} = 2$ variables, we find $z_A\big| = z + \mathrm{i}\zeta\bar{\boldsymbol{\theta}}^2$ which leads to

$$\mathrm{i}\Lambda\big| - \mathrm{i}\check{\Lambda}\big| = \mathrm{i}\Lambda(z + \mathrm{i}\zeta\bar{\boldsymbol{\theta}}^2) - \mathrm{i}\Lambda(z + \mathrm{i}\boldsymbol{\theta}^2/\zeta) = \Lambda'(z)\left(\frac{1}{\zeta}\boldsymbol{\theta}^2 - \zeta\bar{\boldsymbol{\theta}}^2\right) . \tag{B.12}$$

The intrinsic vector multiplet with arbitrary pure gauge A_z is then given by

$$V\big| = \frac{1}{\zeta}\boldsymbol{\theta}^2\left(\frac{\mathrm{i}}{mz} + \Lambda'(z)\right) + \zeta\bar{\boldsymbol{\theta}}^2\left(\frac{\mathrm{i}}{mz} - \Lambda'(z)\right) . \tag{B.13}$$

It remains to verify our claim that V possesses a non-vanishing connection A_z. We begin by observing that the 3D $\mathcal{N} = 4$ spinor derivatives must obey

$$v_i\boldsymbol{\mathcal{D}}^i_\alpha = \boldsymbol{\mathcal{D}}^{(1)}_\alpha = \exp\left(V_{(+)}\hat{e}\right)\boldsymbol{D}^{(1)}_\alpha\exp\left(-V_{(+)}\hat{e}\right) = \boldsymbol{D}^{(1)}_\alpha - \boldsymbol{D}^{(1)}_\alpha V_{(+)}\hat{e} . \tag{B.14}$$

This identifies the spinor U(1) connection as $\boldsymbol{A}^{(1)}_\alpha = \mathrm{i}\boldsymbol{D}^{(1)}_\alpha V_{(+)}$. Note that because $V = V_{(+)} + V_{(-)}$ is analytic, we can just as well denote $\boldsymbol{A}^{(1)}_\alpha = -\mathrm{i}\boldsymbol{D}^{(1)}_\alpha V_{(-)}$, which implies

that $\boldsymbol{A}_\alpha^{(1)} = v_i \boldsymbol{A}_\alpha^i$ is defined everywhere on $\mathbb{C}P^1$. This allows the construction of all the connections. Using (B.2), we note that the analyticity condition $\boldsymbol{D}_\alpha^{(1)} V = 0$ amounts to

$$\boldsymbol{D}_\alpha^2 V_n = \boldsymbol{D}_\alpha^1 V_{n-1} , \qquad \bar{\boldsymbol{D}}_{\underline{2}}^\alpha V_n = -\bar{\boldsymbol{D}}_{\underline{1}}^\alpha V_{n+1} . \tag{B.15}$$

Defining $V_{(+)} := \frac{1}{2} V_0 + \sum_{n=1}^\infty V_n \zeta^n$ it is straightforward to calculate that

$$v_i \boldsymbol{A}_\alpha^i = \mathrm{i} \boldsymbol{D}_\alpha^{(1)} V_{(+)} = \frac{\mathrm{i}}{2} \zeta D_\alpha V_0 + \frac{\mathrm{i}}{2} D_\alpha V_{(-1)} \implies$$

$$\boldsymbol{A}_\alpha^1 = -\frac{\mathrm{i}}{2} \boldsymbol{D}_\alpha^1 V_0 , \qquad \boldsymbol{A}_\alpha^2 = \frac{\mathrm{i}}{2} \boldsymbol{D}_\alpha^1 V_{(-1)} . \tag{B.16}$$

The vector connections can be easily derived from the spinor connections using

$$\{\boldsymbol{\mathcal{D}}_\alpha^1, \boldsymbol{\mathcal{D}}_\beta^2\} + \{\bar{\boldsymbol{\mathcal{D}}}_{\alpha\underline{1}}, \bar{\boldsymbol{\mathcal{D}}}_{\beta\underline{2}}\} = -4\mathrm{i}\varepsilon_{\alpha\beta}\boldsymbol{\mathcal{D}}_z , \tag{B.17a}$$

$$\{\boldsymbol{\mathcal{D}}_\alpha^1, \bar{\boldsymbol{\mathcal{D}}}_{\beta\underline{1}}\} + \{\boldsymbol{\mathcal{D}}_\beta^1, \bar{\boldsymbol{\mathcal{D}}}_{\alpha\underline{1}}\} = -4\mathrm{i}\boldsymbol{\mathcal{D}}_{\alpha\beta} . \tag{B.17b}$$

Now let us specialize to the intrinsic vector multiplet and consider the projection to $\boldsymbol{\theta}_{\underline{2}} = \bar{\boldsymbol{\theta}}^2 = 0$. It is immediately evident that the zero mode $V_0|$ vanishes, and so $\boldsymbol{A}_\alpha^1| = \bar{\boldsymbol{A}}_{\alpha\underline{1}}| = 0$. In other words, from the point of view of the 3D $\mathcal{N} = 2$ superspace, there is no U(1) connection. The remaining connection we need to determine is $\boldsymbol{A}_z$. It is straightforward to calculate

$$\boldsymbol{A}_z = -\frac{1}{8}\Big((\boldsymbol{D}^1)^2 V_{(-1)} - (\bar{\boldsymbol{D}}_{\underline{1}})^2 V_{(1)}\Big) \implies \boldsymbol{A}_z| = \Lambda'(z) . \tag{B.18}$$

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